ON THE STRUCTURE OF SPACES WITH BAKRY-ÉMERY RICCI CURVATURE BOUNDED BELOW

FENG WANG AND XIAOHUA ZHU*

Abstract. In this paper, we explore the structure of limit space for a sequence of Riemannian manifolds with Bakry-Émery Ricci curvature bounded from below in Gromov-Hausdorff topology. We will extend the techniques established by Cheeger and Cloding in [CC1] and [CC2] for Riemannian manifolds with Ricci curvature bounded from below to our case. We prove that each tangent space at a point of the limit space is a metric cone. We also analyze the singular structure of limit space as in [CCT]. Our results will be applied to study the limit space for a sequence of Kähler metrics arising from solutions of certain complex Monge-Ampère equations for the existence problem of Kähler-Ricc soliton on a Fano manifold via the continuity method [TZ1], [TZ2].

Contents

0.	Introduction	1
1.	Volume comparison and the gradient estimate	4
2.	L^2-Integral estimates for Hessians of functions	9
3.	A splitting theorem	15
4.	Existence of metric cone	21
5.	Structure of singular set I: Case of Riemannian metrics	29
6.	Structure of singular set II: Case of Kähler metrics	38
7.	Appendix 1	45
8.	Appendix 2	47
References		53

0. Introduction

In a series of papers [CC1], [CC2], [CC3], Cheeger and Colding studied the singular structure of limit space for a sequence of Riemannian manifolds

1991 *Mathematics Subject Classification.* Primary: 53C25; Secondary: 53C55, 58E11.

Key words and phrases. Bakry-Émery Ricci curvature, Gromov-Hausdorff topology, tangent cone structure.

* Partially supported by the NSFC Grant 10990013 and 11271022.

with Ricci curvature bounded from below in Gromov-Hausdorff topology. As a fundamental result, they proved the existence of metric cone structure for the tangent cone on the limit space [CC2]. Namely,

Theorem 0.1. *([CC2]) Let $(M_i, g_i; p_i)$ be a sequence of n-dimentional Riemannian manifolds which satisfy*

$$Ric_{M_i}(g_i) \geq -(n-1)\Lambda^2 g_i \text{ and } vol_{g_i}(B_{p_i}(1)) \geq v > 0.$$

Then $(M_i, g_i; p_i)$ converge to a metric space $(Y; p_\infty)$ in the pointed Gromov-Hausdorff topology. Moreover, for any $y \in Y$, each tangent cone T_yY is a metric cone over another metric space whose diameter is less than π.

Based on the above theorem, Cheeger and Colding introduced a notion of S_k-typed ($k \leq n-1$) singularities of the limit space Y as follows.

Definition 0.2. *Let $(Y; p_\infty)$ be the limit of $(M_i, g_i; p_i)$ as in Theorem 0.1. We call $y \in (Y; p_\infty)$ a S_k-typed singular point if there exists a tangent cone at y which can be split out an euclidean space $\mathbb{R}^k$ isometrically with dimension at most k.*

As an application of Theorem 0.1 to appropriate tangent cone space T_yY, Cheeger and Colding showed that dimension of set S_k is less than k [CC2]. After that, the important work was made by Cheeger, Colding and Tian [CCT], and Cheeger [C3] to determine which kind of singularities can be excluded in the limit space Y under certain curvature condition for the sequence of (M_i, g_i). For example, Cheeger, Colding and Tian proved

Theorem 0.3. *([CCT]) Let $(M_i, g_i; p_i)$ be a sequence of n-dimensional manifolds and (Y, p_∞) its limit as in Theorem 0.1. Suppose that the integrals*

$$\frac{1}{vol_{g_i}(B_{p_i}(1))} \int_{B_{p_i}(1)} |Rm|^p < \infty$$

are uniformly bounded. Then for any $\epsilon > 0$, the following is true:

$$dim(B_{p_\infty}(1) \setminus R_\epsilon) \leq n-4, \quad \text{if } p = 2$$

and

$$H^{n-2p}(B_{p_\infty}(1) \setminus R_\epsilon) < \infty, \ \text{if } 1 \leq p < 2,$$

where R_ϵ consists of points y in Y which satisfy

$$dist_{GH}(B_y(1), B_0(1)) \leq \epsilon$$

for the unit ball $B_0(1)$ in $\mathbb{R}^n$ and a unit distance ball $B_y(1)$ in some tangent cone T_yY.

The purpose of the present paper is to extend the above Cheeger-Colding theory in Bakry-Emery geometry. Specifically, we analyze the structure of Gromov-Hausdorff limit for a sequence of n-dimentional Riemannian manifolds with Bakry-Emery Ricci curvature bounded below in the class $\mathcal{M}(A,\Lambda,v)$ defined by

$$\begin{aligned}\mathcal{M}(A,\Lambda,v)=\{(M,g;p)|\ & M \text{ is a complete Riemannian manifold which satisfy}\\ & Ric_M(g)+\mathrm{hess}(f)\geq -(n-1)\Lambda^2 g,\\ & \mathrm{vol}_g(B_p(1))\geq v>0, \text{ and } |\nabla f|_g\leq A\}.\end{aligned}$$

Here f is a smooth function on M and $\mathrm{hess}(f)$ denotes Hessian tensor of f. $Ric_M(g)+\mathrm{hess}(f)$ is called the Bakry-Emery Ricci curvature associated to f [BE]. For simplicity, we denote it by $Ric_M^f(g)$. Clearly, $\mathcal{M}(A,\Lambda,v)$ consists of compact Ricci solitons [Ha], [TZh]. We show that both Theorem 0.1 and Theorem 0.3 still hold for a sequence in $\mathcal{M}(A,\Lambda,v)$ (cf. Section 4, 5).

As in [CC1], we shall establish various integral comparison theorems of the gradients and Hessians between appropriate functions and coordinate functions or distance functions on a Riemannian manifold with Bakry-Emery Ricci curvature bounded below. We will use f-harmonic functions to construct those appropriate functions instead of harmonic functions (cf. Section 2). Another technique is to generalize the segment inequality lemmas in [CC1] to our case of weighted volume form (cf. Lemma 3.3, Lemma 3.4, Lemma 3.5) so that Cheeger's triangle lemmas in [C2] are still true on a Riemannian manifold with Bakry-Emery Ricci curvature bounded below (cf. Lemma 3.2, Lemma 4.4). These triangle lemmas are crucial in proofs of the splitting theorem and the metric cone theorem (cf. Theorem 3.1, Theorem 4.3).

Another motivation of this paper is to study the limit space for a sequence of Kähler metrics g_{t_i} arising from solutions of certain complex Monge-Ampère equations for the existence problem of Kähler-Ricci soliton via the continuity method [TZ1], [TZ2]. We show that such metrics are naturally belonged to $\mathcal{M}(A,v,\Lambda)$. As a consequence, for any sequence $\{g_{t_i}\}$ there exists a subsequence which converges to a metric space with complex codimention of singularities at least one in Gromov-Hausdorff topology (cf. Theorem 6.2, Section 6). Furthermore, in addition of assumption that such metrics are weak almost Kähler-Ricc solitons (cf. Theorem 6.7), we prove

Theorem 0.4. *Suppose that $\{g_{t_i}\}$ is a sequence of weak almost Kähler-Ricci solitons on a Fano manifold M. Then there exists a subsequence of $\{g_{t_i}\}$ which converges to a metric space (M_∞, g_∞) in the Gromov-Hausdorff topology. The complex codimension of singularities of M_∞ is at least two.*

As a corollary of Theorem 0.4, we show that there exists a sequence of weak almost Kähler-Ricc solitons on M which converges to a metric space (M_∞, g_∞) with complex codimension of the singular part of (M_∞, g_∞) at least two in Gromov-Hausdorff topology if the modified Mabuchi K-energy defined in [TZ1] is bounded from below. In a sequel of paper, we will further confirm that the regular part of (M_∞, g_∞) is in fact a Kähler-Ricc soliton by using Ricci flow method as done in a recent paper by Wang and Tian [WT].

The organization of paper is as follows. In Section 1, we first recall the volume comparison theorem by Wei and Wylie. Then we prove a version of Li-Yau typed gradient estimate for f-harmonic functions when Bakry-Émery Ricci curvature is bounded from below. In Section 2, we give various integral estimates for gradients and Hessians of f-harmonic functions. In Section 3 and Section 4, we will prove the splitting theorem (cf. Theorm 3.1) and the metric cone theorem (cf. Theorem 4.3), respectively. In Section 5, we give a generalization of Cheeger-Colding-Tian's Theorem 0.3 in the setting of Bakry-Émery geometry (cf. Theorem 5.4). In Section 6, we prove Theorem 6.2 and Theorem 6.7. Section 7 is an appendix where we explain how to use the technique of conformal transformation in [TZh] to give another proof of Theorem 6.2 and Theorem 6.3. Section 8 is another appendix where (6.10) in Section 6 is proved.

Acknowledgements. The authors would like to thank Professor Gang Tian for many valuable discussions.

1. Volume comparison and the gradient estimate

The notion of Bakry-Emery Ricci curvature $Ric^f_M(g)$ associated to a smooth function f on a Riemannian manifold (M, g) was first appeared in [BE]. Related to the conformal geometry, one can introduce a weighted volume form and a f-Lapalace operator associated to f on (M, g) as follows,

$$d\mathrm{v}^f = e^{-f} d\mathrm{v} \text{ and } \Delta^f = \Delta - \langle \nabla f, \nabla \rangle.$$

Then Δ^f can be extended a self-adjoint elliptic operator under the following weighted inner product,

$$(u, v) = \int_M uv d\mathrm{v}^f, \ \forall\ u, v \in L^2(M).$$

That is

$$\int_M \Delta^f uv d\mathrm{v}^f = \int_M \langle \nabla u, \nabla v \rangle d\mathrm{v}^f = \int_M \Delta^f vu d\mathrm{v}^f.$$

The divergence theorem with respect to Δ^f is

$$\int_\Omega \Delta^f u d\mathrm{v}^f = \int_{\partial\Omega} \langle \nabla u, n \rangle e^{-f} d\sigma,$$

where Ω is a domain in M with piece-wise smooth boundary, n denotes the outer unit normal vector field on $\partial\Omega$ and $d\sigma$ is an induced area form of g on $\partial\Omega$.

Let $r = r(q) = \mathrm{dist}(p, q)$ be a distance function on (M, g). In [WW], Wei and Wylie computed f-Laplacian for r and got the following comparison result under Bakry-Emery Ricci curvature.

Lemma 1.1. *([WW]) Let (M, g) be a complete Riemannian manifold which satisfies*

$$Ric^f_M(g) \geq -(n-1)\Lambda^2 g. \tag{1.1}$$

Then

$$\Delta^f r \leq (n-1+4A)\Lambda \coth \Lambda r, \quad \textit{if } |f| \leq A, \tag{1.2}$$

and

$$\Delta^f r \leq (n-1)\Lambda \coth \Lambda r + A, \quad \textit{if } |\nabla f| \leq A. \tag{1.3}$$

As an application of Lemma 1.1, Wei and Wylie proved the following weighted volume comparison theorem.

Theorem 1.2. *([WW]) Let (M, g) be a complete Riemannian manifold which satisfies (1.1). Then for any $0 < r \leq R$,*

$$\frac{vol^f(B_p(r))}{vol^f(B_p(R))} \geq \frac{vol_\Lambda^{n+4A}(B(r))}{vol_\Lambda^{n+4A}(B(R))}, \quad \textit{if} \ \ |f| \leq A, \tag{1.4}$$

and

$$\frac{vol^f(B_p(r))}{vol^f(B_p(R))} \geq e^{-AR}\frac{vol_\Lambda^n(B(r))}{vol_\Lambda^n(B(R))}, \ \textit{if } |\nabla f| \leq A, \tag{1.5}$$

where $vol_\Lambda^m(B(r))$ denotes the volume of geodesic ball $B(r)$ with radius r in m-dimensional form space with constant curvature $-\Lambda$.

The proof of Theorem 1.2 depends on a monotonic formula for the weighted volume form as follows.

By choosing a polar coordinate with the origin at p, we write

$$e^{-f}d\mathrm{v} = A^f(s,\theta)ds \wedge d\theta.$$

Then

$$\frac{d}{ds}A^f(s,\theta) = A^f(s,\theta)\Delta^f r.$$

In case that $|\nabla f| \leq A$, it follows from (1.3),

$$\frac{d}{ds}A^f(s,\theta) \leq A^f(s,\theta) l_{\Lambda,A}(r), \tag{1.6}$$

where $l_{\Lambda,A}(r) = (n-1)\Lambda \coth \Lambda r + A$. Thus if we put

$$L_{\Lambda,A}(r) = e^{Ar}(\frac{\sinh \Lambda r}{\Lambda})^{n-1}, \tag{1.7}$$

which is a solution of equation,

$$\frac{L'_{\Lambda,A}}{L_{\Lambda,A}} = l_{\Lambda,A}, \ \frac{L_{\Lambda,A}(r)}{r^{n-1}} \to 1 \text{ as } r \to 0, \tag{1.8}$$

(1.6) is equivalent to the following monotonic formula,

$$\frac{A^f(b,\theta)}{A^f(a,\theta)} \le \frac{L_{\Lambda,A}(b)}{L_{\Lambda,A}(a)}, \ \forall\ b \ge a. \tag{1.9}$$

By a simple computation, we get (1.5) from (1.9). Similarly, we can obtain (1.4).

Another important formula is the so-called Bochner-typed identity,

$$\begin{aligned} &\frac{1}{2}\Delta^f |\nabla u|^2 \\ &= |\text{hess } u|^2 + \langle \nabla u, \nabla \Delta^f u\rangle + Ric_g^f(\nabla u, \nabla u), \ \forall\ u \in C^\infty(M). \end{aligned} \tag{1.10}$$

By (1.10) and Lemma 1.1, we derive the following Li-Yau typed gradient estimate for f- harmonic functions on (M, g).

Proposition 1.3. *Let $u > 0$ be a f-harmonic function defined on the unit distance ball $B_p(1) \subset (M, g)$, i.e.*

$$\triangle^f u = 0, \ \textit{in} \ \ B_p(1).$$

Suppose that

$$Ric_g^f(M) \ge -(n-1)\Lambda^2 g, \ \textit{and } |\nabla f| \le A. \tag{1.11}$$

Then

$$|\nabla u|^2 \le (C_1\Lambda + C_2A^2 + C_3)u^2, \ \textit{in } B_p(1/2), \tag{1.12}$$

where the constants C_i $(1 \le i \le 3)$ depend only on n.

Proof. The proof is standard as in the case $f = 0$ for a harmonic function (cf. [SY]). In general, let $v = \ln u$. Then

$$\begin{aligned} \Delta^f v &= \Delta v - \langle \nabla f, \nabla v\rangle = \nabla(\frac{\nabla u}{u}) - \langle \nabla f, \frac{\nabla u}{u}\rangle \\ &= \frac{\Delta u}{u} - \frac{|\nabla u|^2}{u^2} - \langle \nabla f, \frac{\nabla u}{u}\rangle = \frac{|\nabla u|^2}{u^2}. \end{aligned} \tag{1.13}$$

Note that

$$|\text{hess } v|^2 \ge \frac{|\Delta v|^2}{n}$$

and

$$|\Delta v|^2 \ge \frac{|\Delta^f v|^2}{2} - C_1 A^2 Q,$$

where $Q = |\nabla v|^2$. Thus applying (1.10) to v, we get

$$\frac{1}{2}\Delta^f Q \geq \frac{Q^2}{2n} - \frac{1}{n}C_1 A^2 Q + \langle \nabla v, \nabla Q\rangle - \Lambda^2 Q. \tag{1.14}$$

Choose a decreasing cut-off function $\eta(t)$ on $t \in [0,\infty]$ such that

$$\begin{aligned} &\eta(t) = 1 \text{ if } t \leq \frac{1}{2}; \phi = 0 \ \text{ if } t \geq 1;\\ &-C_2\eta^{\frac{1}{2}} \leq \eta', \text{ if } t \geq \frac{1}{2};\\ &|\eta''| \leq C_2. \end{aligned}$$

Then if let $\phi = \eta(r(\cdot, p))$,

$$|\nabla\phi|^2\phi^{-1} \leq C_2^2,$$

and by Lemma 1.1,

$$\Delta^f\phi = \Delta^f\eta(r) \geq -C_3(A + \Lambda). \tag{1.15}$$

Hence, by (1.14), we obtain

$$\begin{aligned} \Delta^f(\phi Q) &= \Delta(\phi Q) + \langle \nabla f, \nabla(\phi Q)\rangle\\ &= \phi\Delta^f Q + Q\Delta^f\phi + 2\langle\nabla Q, \nabla\phi\rangle\\ &\geq \phi(\frac{Q^2}{n} - (\frac{2}{n}C_1A^2 + 2\Lambda^2)Q) - C_3(A+\Lambda)Q\\ &+ 2\langle\nabla v, \nabla Q\rangle + 2\langle\nabla Q, \nabla\phi\rangle. \end{aligned} \tag{1.16}$$

Suppose that $(Q\phi)(q) = \max_M\{Q\phi\}$ for some $q \in M$. Then at this point, it holds $\nabla(Q\phi) = 0$. It follows that

$$\nabla Q = -\frac{Q\nabla\phi}{\phi},$$

and

$$|\langle\nabla Q, \nabla\phi\rangle| = \frac{Q}{\phi}|\nabla\phi|^2 \leq C_2^2 Q.$$

Also

$$|\langle\nabla Q, \nabla v\rangle| \leq Q^{\frac{3}{2}}\frac{|\nabla\phi|}{\phi} \leq C_2Q^{\frac{3}{2}}\phi^{-\frac{1}{2}}.$$

Therefore, by applying maximum principle to ϕQ at the point q, we get from (1.16),

$$\begin{aligned} 0 \geq\ &\phi(\frac{Q^2}{n} - \frac{2}{n}C_1A^2Q - 2C_2Q^{\frac{3}{2}}\phi^{-\frac{1}{2}})\\ &- C_3(\Lambda Q + A) - 2C_2Q. \end{aligned}$$

As a consequence, we derive

$$\phi Q \leq (\phi Q)(q) \leq C_4\Lambda + C_5A^2 + C_6, \text{ on } B_p(1).$$

This proves the proposition. □

As an application of Proposition 1.3, we are able to construct a cut-off function with bounded gradient and f-Lapalace. Such a function will be used in the next section.

Lemma 1.4. *Under the condition (1.11) in Proposition 1.3, there exists a cut-off function ϕ supported in $B_p(2)$ such that i) $\phi \equiv 1$, in $B_p(1)$; ii)*

$$|\nabla\phi|, |\Delta^f\phi| \leq C(n, \Lambda, A).$$

Proof. We will use an argument from Theorem 6.33 in [CC1]. First we consider a solution of ODE,

$$G'' + G'l = 1, \text{ on } [1, 2], \tag{1.17}$$

with $G(1) = a$ and $G(2) = 0$. It is easy to see that there is a number $a = a(n, \Lambda, A)$ such tha $G' < 0$. Then by (1.3), we have

$$\Delta^f G(d(., p)) \geq 1.$$

Let w be a solution of equation,

$$\Delta^f w = \frac{1}{a}, \text{ in } B_p(2) \setminus \overline{B_p(1)},$$

with $w = 1$ on $\partial B_p(1)$ and $w = 0$ on $\partial B_p(2)$. Thus by the maximum principle, we get

$$w \geq \frac{G(d(., p))}{a}.$$

Secondly, we choose another function H with $H' > 0$ which is a solution of ODE,

$$H'' + H'l = 1, \quad \text{on } [0, \infty), \tag{1.18}$$

with $H(0) = 0$. Then by (1.3), we have

$$\Delta^f H(d(., x)) \leq 1, \text{ for any fixed point } x.$$

Thus by the maximum principle, we get

$$w(y) - \frac{H(d(y, x))}{a} \leq max\{1 - \frac{H(d(x, p) - 1)}{a}, 0\}$$

for any y in the annulus $A_p(1, 2) = B_p(2) \setminus \overline{B_p(1)}$. It follows

$$w(x) \leq max\{1 - \frac{H(d(x, p) - 1)}{a}, 0\}, \ \forall\ x \in A_p(1, 2).$$

Now we choose a number $\eta(n, \Lambda, A)$ such that $\frac{G(1+\eta)}{a} > 1 - \frac{H(1-\eta)}{a}$ and we define a function $\psi(x)$ on $[0, 1]$ with bounded derivative up to second order, which satisfies

$$\psi(x) = 1, \text{ if } x \geq \frac{G(1 + \eta)}{a}$$

and

$$\psi(x) = 0, \text{ if } x \le max\{1 - \frac{H(1-\eta)}{a}, 0\}.$$

It is clear that $\phi = \psi \circ w$ is constant near the boundary of $A_p(1,2)$. So we can extend ϕ inside $B_p(1)$ by setting $\phi = 1$. By Proposition 1.3, one sees that $|\nabla \phi|$ is bounded by a constant $C(n, \Lambda, A)$ in $B_2(p)$. Since

$$\Delta^f \phi = \psi'' |\nabla w|^2 + \psi' \Delta^f w,$$

we also derive that $|\Delta^f \phi| \le C(n, \Lambda, A)$. □

2. L^2-Integral estimates for Hessians of functions

In this section, we establish various integral comparisons of gradient and Hessian between appropriate f-harmonic functions and coordinate functions or distance functions. We start with a basic lemma about a distance function along a long approximate line in a manifold.

Lemma 2.1. *Let (M, g) be a complete Riemannian manifold which satisfies*

$$Ric_M^f(g) \ge -\frac{n-1}{R^2} g \text{ and } |f| \le A. \tag{2.1}$$

Suppose that there are three points p, q^+, q^- in M which satisfy

$$d(p,q^+) + d(p,q^-) - d(q^+,q^-) < \epsilon \tag{2.2}$$

and

$$d(p,q^+), d(p,q^-) > R. \tag{2.3}$$

Then for any $q \in B_p(1)$, the following holds,

$$E(q) := d(q,q^+) + d(q,q^-) - d(q^+,q^-) < \Psi(\epsilon, \frac{1}{R}; A, n),$$

where the quantity $\Psi(\epsilon, \frac{1}{R}; A, n)$ means that it goes to zero as $\epsilon, \frac{1}{R}$ go to zero while A, n are fixed.

Proof. Let

$$\tilde{l}(s) = (n - 1 + 4A)\frac{1}{R} \coth \frac{s}{R}. \tag{2.4}$$

For given $t > 0$, we construct a function $G = G_t(s)$ on $[0,t]$ which satisfies the ODE,

$$G'' + \tilde{l}(s)G' = 1, G'(s) < 0, \tag{2.5}$$

with $G(0) = +\infty$ and $G(t) = 0$. Then $G(s) \sim s^{2-n-4A}$ $(s \to 0)$. Furthermore, by (1.2) in Lemma 1.1, we have

$$\Delta^f G(d(x,.)) = G'\Delta^f d(x,.) + G'' \ge G'' + G'\tilde{l}(s) = 1. \tag{2.6}$$

By Lemma 1.1,

$$\Delta^f E(q) \leq \frac{10(n-1+A)}{R} := b. \tag{2.7}$$

Claim 2.2. *For any* $0 < c < 1$,

$$E(q) \leq 2c + bG_1(c) + \epsilon, \quad \text{if } bG_1(c) > \epsilon.$$

Suppose that the claim is not true. Then there exists point $q_0 \in B_p(1)$ such that for some c,

$$E(q_0) > 2c + bG_1(c) + \epsilon.$$

We consider $u(x) = bG_1(d(q_0, x)) - E(x)$ in the annulus $A_{q_0}(c, 1)$. Clearly,

$$\Delta^f u \geq 0.$$

Note that we may assume that $p \in A_{q_0}(c, 1)$. Otherwise $d(q_0, p) < c$ and $E(q_0) \leq E(p) + 2c$, so the claim is true and the proof is complete. On the other hand, it is easy to see that on the inner boundary $\partial B_{q_0}(c)$,

$$u(x) = bG_1(c) - E(x) \leq bG_1(c) - E(q_0) - 2c \leq -\epsilon,$$

and on the outer boundary $\partial B_q(1)$,

$$u(x) = -E(x) \leq 0.$$

Thus applying the maximum principle, it follows that $u(p) \leq 0$. However,

$$u(p) = bG_1(d(p, q_0)) - E(p) \geq bG_1(c) - \epsilon,$$

which is impossible. Therefore, the claim is true.

By choosing c with the order $b^{\frac{1}{n-1+4A}}$ in Claim 2.2, we prove Lemma 2.1.

□

Let $b^+(x) = d(q^+, x) - d(q^+, p)$ and let h^+ be a f-harmonic function which satisfies

$$\triangle^f h^+ = 0, \text{ in } B_p(1),$$

with $h^+ = b^+$ on $\partial B_p(1)$. Then

Lemma 2.3. *Under the conditions in Lemma 2.1 with* $|\nabla f| \leq A$, *we have*

$$|h^+ - b^+|_{C^0(B_p(1))} < \Psi(1/R, \epsilon; A), \tag{2.8}$$

$$\frac{1}{vol(B_p(1))} \int_{B_p(1)} |\nabla h^+ - \nabla b^+|^2 e^{-f} dv < \Psi(1/R, \epsilon; A), \tag{2.9}$$

$$\frac{1}{vol(B_p(\frac{1}{2}))} \int_{B_p(\frac{1}{2})} |hess\ h^+|^2 e^{-f} dv < \Psi(1/R, \epsilon; A). \tag{2.10}$$

Proof. Choose a point q in $\partial B_p(2)$ and let $g = \varphi(d(q,\cdot))$, where φ is a solution of (2.5) restricted on the interval $[1,3]$. Then

$$\Delta^f g = \varphi'\Delta^f r + \varphi'' \geq \varphi' l + \varphi'' = 1, \text{ in } B_p(1). \tag{2.11}$$

It follows that

$$\Delta^f(h^+ - b^+ + \Psi(1/R,\epsilon;A)g) > 0, \text{ in } B_p(1).$$

Thus by the maximum principle, we get

$$h^+ - b^+ < \Psi(1/R,\epsilon;A).$$

On the other hand, we have

$$\Delta^f(-b^- - h^+ + \Psi(1/R,\epsilon;A)g) > 0, \text{ in } B_p(1),$$

where $b^- = d(q^-,x) - d(p,q^-)$. Since $b^+ + b^-$ is small as long as $1/R$ and ϵ are small by Lemma 2.1, by the maximum principle, we also get

$$h^+ - b^+ > -(b^+ + b^-) - \Psi(1/R,\epsilon;A) > -\Psi(1/R,\epsilon;A).$$

For the second estimate (2.9), we see

$$\begin{aligned}&\int_{B_p(1)} |\nabla h^+ - \nabla b^+|^2 e^{-f} d\mathrm{v}\\ &= \int_{B_p(1)} (h^+ - b^+)(\triangle^f b^+ - \triangle^f h^+) e^{-f} d\mathrm{v}\\ &< \Psi(1/R,\epsilon;A)\int_{B_p(1)} |\triangle^f b^+| e^{-f} d\mathrm{v}.\end{aligned}$$

and

$$\begin{aligned}&\int_{B_p(1)} |\triangle^f b^+| e^{-f} d\mathrm{v}\\ &\leq |\int_{B_p(1)} \triangle_f b^+ e^{-f} d\mathrm{v}| + 2e^A \sup_{B_p(1)}(\triangle^f b^+)\mathrm{vol}(B_p(1))\\ &\leq e^A \mathrm{vol}(\partial B_p(1)) + C(A)\mathrm{vol}(B_p(1))\\ &\leq C'(A)\mathrm{vol}(B_p(1)).\end{aligned}$$

Here we used (1.9) at the last inequality. Then (2.9) follows.

To get (2.10), we choose a cut-off function φ supported in $B_p(1)$ as constructed in Lemma 1.4. Since

$$\begin{aligned}&\Delta^f(|\nabla h^+|^2 - |\nabla b^+|^2)\\ &= |\text{hess } h^+|^2 + Ric^f(\nabla h^+, \nabla h^+),\end{aligned}$$

multiplying both sides of the above by $\varphi e^{-f} d\mathrm{v}$ and using integration by parts, we get

$$\int_{B_p(1)} \varphi |\mathrm{Hess}\ h^+|^2 e^{-f} d\mathrm{v}$$
$$\leq \int_{B_p(1)} \Delta^f \varphi (|\nabla h^+|^2 - |\nabla b^+|^2) e^{-f} d\mathrm{v} + \frac{n-1}{R^2} \int_{B_p(1)} \varphi |\nabla h^+|^2 e^{-f} d\mathrm{v}.$$

Note that $|\nabla h^+|$ is locally bounded by Proposition 1.3, we derive (2.10) from (2.9) immediately. ☐

Next, we construct an approximate function to compare the sqaure of a distance function with asymptotic integral gradient and Hessian estimates. Such estimates are crucial in the proof of metric-cone theorem in Section 4.

Let $q \in M$ and h be a solution of the following equation,

$$\Delta^f h = n, \text{ in } B_q(b) \setminus \overline{B_p(a)},\ h|\partial B_q(b) = \frac{b^2}{2} \text{ and } h|\partial B_q(a) = \frac{a^2}{2}. \tag{2.12}$$

Let $p = \frac{r(q,\cdot)^2}{2}$. Then

Lemma 2.4. *Let (M, g) be a complete Riemannian manifold which satisfies*

$$Ric^f_M(g) \geq -(n-1)\epsilon^2 \Lambda^2 g \text{ and } |\nabla f| \leq \epsilon A.$$

Let $a < b$. Suppose that

$$\frac{vol^f(\partial B_q(b))}{vol^f(\partial B_q(a))} \geq (1-\omega) \frac{L_{\epsilon\Lambda,\epsilon A}(b)}{L_{\epsilon\Lambda,\epsilon A}(a)} \tag{2.13}$$

for some $\omega > 0$, where $L_{\epsilon\Lambda,\epsilon A}(r)$ is the function defined by (1.7) with respect to the constants $\epsilon\Lambda$ and ϵA. Then

$$\frac{1}{vol(A(a,b))} \int_{A_q(a,b)} |\nabla p - \nabla h|^2 e^{-f} dv < \Psi(\omega, \epsilon; \Lambda, A, a, b), \tag{2.14}$$

where $A_q(a,b) = B_q(b) \setminus \overline{B_p(a)}$.

Proof. Since

$$\Delta^f r \leq (n-1)\epsilon\Lambda \coth(\epsilon\Lambda r) + \epsilon A = l_{\epsilon\Lambda,\epsilon A},$$

we have

$$\Delta^f p = p'' + p' \Delta^f r < n + \Psi(\epsilon; \Lambda, A, a, b), \quad \text{in } A(a,b). \tag{2.15}$$

Thus we get

$$\frac{1}{\mathrm{vol}(A_q(a,b))} \int_{A(a,b)} \Delta^f p e^{-f} d\mathrm{v} < e^{-f(0)}(n + \Psi(\epsilon; \Lambda, A, a, b)). \tag{2.16}$$

On the other hand, by the monotonicity formula (1.9), we have

$$\frac{\int_a^b L_{\epsilon\Lambda,\epsilon A}(s)ds}{L_{\epsilon\Lambda,\epsilon A}(b)}\mathrm{vol}^f(\partial B_q(b)) \le \mathrm{vol}^f(A_q(a,b)) \le \frac{\int_a^b L_{\epsilon\Lambda,\epsilon A}(s)ds}{L_{\epsilon\Lambda,\epsilon A}(a)}\mathrm{vol}^f(\partial B_q(a)).$$

It follows by (2.13),

$$\mathrm{vol}^f(A_q(a,b)) \le (1-\omega)^{-1}\frac{\int_a^b L_{\epsilon\Lambda,\epsilon A}(s)ds}{L_{\epsilon\Lambda,\epsilon A}(b)}\mathrm{vol}^f(\partial B_q(b)).$$

Since

$$\int_{A_q(a,b)} \Delta^f p e^{-f} d\mathrm{v} = b\mathrm{vol}^f(\partial B_q(b)) - a\mathrm{vol}^f(\partial B_q(a)),$$

we get

$$\begin{aligned}&\frac{1}{\mathrm{vol}^f(A_q(a,b))}\int_{A(a,b)} \Delta^f p e^{-f} d\mathrm{v}\\ &\ge (1-\omega)\frac{L_{\epsilon\Lambda,\epsilon A}(b)}{\int_a^b L_{\epsilon\Lambda,\epsilon A}(s)ds}(b - a\frac{\mathrm{vol}^f(\partial B_q(a))}{\mathrm{vol}^f(\partial B_q(b))}).\end{aligned}$$

Observe that vol^f is close to $e^{-f(0)}\mathrm{vol}$ and $\frac{L_{\epsilon\Lambda,\epsilon A}(s)}{s^{n-1}}$ is close to a constant as ϵ is small. Hence we derive immediately,

$$\frac{1}{\mathrm{vol}(A_q(a,b))}\int_{A_q(a,b)} \Delta^f p e^{-f} d\mathrm{v} > e^{-f(0)}(n + \Psi(\omega,\epsilon;\Lambda,A,a,b)). \tag{2.17}$$

By (2.16) and (2.17), we have

$$|\int_{A_q(a,b)} (\Delta^f p - n)e^{-f} d\mathrm{v}| < \mathrm{vol}(A_q(a,b))\Psi(\omega,\epsilon;\Lambda,A,a,b).$$

Then one can follow the argument for the estimate (2.9) in Lemma 2.3 to obtain (2.14).

□

Furthermore, we have

Lemma 2.5. *Under the condition in Lemma 2.4, it holds*

$$\begin{aligned}&\frac{1}{vol(A_q(a_2,b_2))}\int_{A(a_2,b_2)} |hess\ h - g|^2 e^{-f} dv\\ &< \Psi(\omega,\epsilon;\Lambda,A,a_1,b_1,a_2,b_2,a,b),\end{aligned} \tag{2.18}$$

where $a < a_1 < a_2 < b_2 < b_1 < b$.

Proof. First observe that

$$\begin{aligned}&\frac{1}{\mathrm{vol}(A_q(a,b))}\int_{A_q(a,b)} |\mathrm{hess}\ h - g|^2 e^{-f} d\mathrm{v}\\ &= \frac{1}{\mathrm{vol}(A_q(a,b))}\int_{A_q(a,b)} |\mathrm{hess\ h}|^2 e^{-f} d\mathrm{v} + \frac{1}{\mathrm{vol}(A_q(a,b))}\int_{A_q(a,b)} (n - 2\Delta h)e^{-f}\mathrm{v}.\end{aligned}$$

Let φ be a cut-off function of $A_q(a,b)$ with support in $A_q(a_1,b_1)$ as constructed in Lemma 1.4 which satisfies,

$$1)\ \varphi\equiv 1,\quad \text{in } A_q(a_2,b_2);$$
$$2)\ |\nabla\varphi|, |\triangle^f\varphi| \ \text{ is bounded in } A_q(a,b).$$

Then

$$\begin{aligned}
&\frac{1}{\mathrm{vol}(A_q(a,b))}\int_{A_q(a,b)}\varphi|\text{hess h}-g|^2 e^{-f}d\mathrm{v}\\
&=\frac{1}{\mathrm{vol}(A_q(a,b))}\int_{A_q(a,b)}\varphi|\text{hess } h|^2 e^{-f}d\mathrm{v}\\
&+\frac{1}{\mathrm{vol}(A_q(a,b))}\int_{A_q(a,b)}\varphi(n-2\Delta h)e^{-f}d\mathrm{v}.
\end{aligned}\tag{2.19}$$

By the Bochner formula (1.10), we have

$$\begin{aligned}
&\frac{1}{\mathrm{vol}(A_q(a,b))}\int_{A_q(a,b)}\varphi|\text{hess } h|^2 e^{-f}d\mathrm{v}\\
&<\frac{1}{2\mathrm{vol}(A_q(a,b))}\int_{A_q(a,b)}\varphi\Delta^f|\nabla h|^2 e^{-f}d\mathrm{v}+\Psi(\epsilon;\Lambda,A,a_1,b_1,a_2,b_2,a,b).
\end{aligned}$$

It follows by Lemma 2.4,

$$\begin{aligned}
&\frac{1}{\mathrm{vol}(A_q(a,b))}\int_{A_q(a,b)}\varphi|\text{hess } h|^2 e^{-f}d\mathrm{v}\\
&<\frac{1}{2\mathrm{vol}(A_q(a,b))}\int_{A_q(a,b)}\varphi\Delta^f|\nabla p|^2 e^{-f}d\mathrm{v}+\Psi(\epsilon,\omega;\Lambda,A,a_1,b_1,a_2,b_2,a,b)\\
&=\frac{1}{\mathrm{vol}(A_q(a,b))}\int_{A_q(a,b)}\varphi\Delta^f p e^{-f}d\mathrm{v}+\Psi(\epsilon,\omega;\Lambda,A,a_1,b_1,a_2,b_2,a,b).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\frac{1}{\mathrm{vol}(A_q(a,b))}\int_{A_q(a,b)}\varphi(n-2\Delta h)e^{-f}d\mathrm{v}\\
&=\frac{1}{\mathrm{vol}(A_q(a,b))}\int_{A_q(a,b)}\varphi(-n-2\langle\nabla f,\nabla h\rangle)e^{-f}d\mathrm{v}\\
&=\frac{1}{\mathrm{vol}(A_q(a,b))}\int_{A_q(a,b)}-n\varphi e^{-f}d\mathrm{v}+\Psi(\epsilon,\omega;\Lambda,A,a_1,b_1,a,b).
\end{aligned}$$

Hence we derive from (2.19),

$$\begin{aligned}
&\frac{1}{\mathrm{vol}(A(a_2,b_2))}\int_{A_q(a_2,b_2)}|\mathrm{hess}\ h-g|^2e^{-f}d\mathrm{v}\\
&\le \frac{1}{\mathrm{vol}(A_q(a,b))}\int_{A_q(a,b)}\varphi|\mathrm{hess}\ h-g|^2e^{-f}d\mathrm{v}\\
&< \frac{1}{\mathrm{vol}(A_q(a,b))}\int_{A_q(a,b)}\varphi(\Delta^f p-n)e^{-f}d\mathrm{v}+\Psi(\epsilon,\omega;\Lambda,A,a_1,b_1,a_2,b_2,a,b)\\
&< \Psi(\epsilon,\omega;\Lambda,A,a_1,b_1,a_2,b_2,a,b).
\end{aligned}$$

Here we used (2.15) at last inequality. □

3. A splitting theorem

In this section, we prove the splitting theorem of Cheeger-Colding in Bakry-Émery geometry [CC1]. Recall that $\gamma(t)$ ($t\in(-\infty,\infty)$) is a line in a metric space Y if

$$\mathrm{dist}(\gamma(t_1),\gamma(t_2))=|t_1-t_2|,\ \forall\ t_1,t_2\in(-\infty,\infty).$$

Theorem 3.1. *Let $(M_i,g_i;p_i)$ be a sequence of Riemannian manifolds which satisfy*

$$Ric_{M_i}^{f_i}(g_i)\ge-\epsilon_i^2g_i,\ \ |f_i|,\ \ |\nabla f_i|\le A.$$

Let $(Y;y)$ be a limit metric space of $(M_i,g_i;p_i)$ in the pointed Gromov-Hausdorff topology as $\epsilon_i\to 0$. Suppose that Y contains a line passing y. Then $Y=\mathbb{R}\times X$ for some metric space X.

We will follow the argument in [CC1] to prove Theorem 3.1. First we recall a Cheeger's triangular lemma in terms of small integral Hessian of appropriate function.

Lemma 3.2. *(Lemma 9.16, [C1]) Let x,y,z be three points in a complete Riemannian manifold M. Let $\gamma(s)$ ($s\in[0,a]$, $a=d(x,y)$) be a geodesic curve connecting x,y and $\gamma_s(t)$ ($s\in[0,l(s)]$, $l(s)=d(z,\gamma(s))$) a family of geodesic curves connecting z and $\gamma(s)$. Suppose that h is a smooth function on M which satisfies*

$$i)\ |h(z)-h(x)|<\delta<<1,$$

$$ii)\ \int_{[0,a]}|\nabla h(\gamma(s))-\gamma'(s)|<\delta<<1,$$

$$iii)\ \int_{[0,a]}\int_{[0,l(s)]}|hess\ h(\gamma_s(t))|dtds<\delta<<1.$$

Then

$$|d(x,y)^2+d(x,z)^2-d(y,z)^2|<\epsilon(\delta)<<1. \tag{3.1}$$

In order to get the above configuration in Lemma 3.2, we need a segment inequality lemma in terms of Bakry-Émery Ricci curvature. In the following, we will always assume that the manifold (M, g) satisfies

$$Ric_M^f(g) \geq -(n-1)\Lambda^2, \ |f|, |\nabla f| \leq A, \tag{3.2}$$

and the volume form $d\mathrm{v}$ is replaced by $d\mathrm{v}^f = e^{-f}d\mathrm{v}$.

Lemma 3.3. *Let A_1, A_2 be two subsets of M and W another subset of M such that $\bigcup_{y_1 \in A_1, y_2 \in A_2} \gamma_{y_1 y_2} \subseteq W$, where $\gamma_{y_1 y_2}$ is a minimal geodesic curve connecting y_1 and y_2 in M. Let*

$$D = sup\{d(y_1, y_2)|\ y_1 \in A_1, y_2 \in A_2\}.$$

Then for any smooth function e on W, it holds

$$\begin{aligned} &\int_{A_1 \times A_2} \int_0^{d(y_1,y_2)} e(\gamma_{y_1,y_2}(s))ds \\ &\leq c(n, \Lambda, A)D[vol^f(A_1) + vol^f(A_2)] \int_W e, \end{aligned} \tag{3.3}$$

where $c(n, \Lambda, A) = sup\{L_{\Lambda,A}(s)/L_{\Lambda,A}(u)|\ 0 < \frac{s}{2} \leq u \leq s\}$.

Proof. Note that

$$\begin{aligned} &\int_{A_1 \times A_2} \int_0^{d(y_1,y_2)} e(\gamma_{y_1,y_2}(s))ds \\ &= \int_{A_1} dy_1 \int_{A_2} \int_{\frac{d(y_1,y_2)}{2}}^{d(y_1,y_2)} e(\gamma_{y_1 y_2}(s))dsdy_2 \\ &+ \int_{A_2} dy_2 \int_{A_1} \int_{\frac{d(y_1,y_2)}{2}}^{d(y_1,y_2)} e(\gamma_{y_1 y_2}(s))dsdy_1. \end{aligned}$$

On the other hand, for a fixed $y_1 \in A_1$, by using the monotonicity formula (1.9), we have

$$\begin{aligned} &\int_{A_2} \int_{\frac{d(y_1,y_2)}{2}}^{d(y_1,y_2)} e(\gamma_{y_1 y_2}(s))dsdy_2 \\ &= \int_{A_2} \int_{\frac{r}{2}}^{r} e(\gamma_{y_1 y_2}(s))A^f(r,\theta)drd\theta ds \\ &\leq c(n, \Lambda, A) \int_{A_2} \int_{\frac{r}{2}}^{r} e(\gamma_{y_1 y_2}(s))A^f(s,\theta)drd\theta ds \\ &\leq c(n, \Lambda, A)D \int_W e. \end{aligned}$$

Similarly,

$$\begin{aligned}&\int_{A_1}\int_{\frac{d(y_1,y_2)}{2}}^{d(y_1,y_2)} e(\gamma_{y_1y_2}(s))dsdy_1\\&\leq c(n,\Lambda,A)D\int_W e.\end{aligned}$$

Then (3.3) follows from the above two inequalities.

□

Using the same argument above, we can prove

Lemma 3.4. *Given two points q^-, q with $d(q,q^-) > 10$ and a smooth function e with support in $B_p(1)$, the following inequality holds,*

$$\int_{B_q(r)} dy\int_0^{d(q^-,y)} e(\gamma_{q^-y}(s))ds \leq c(\Lambda,A)\int_{B_p(1)} e(y)dy. \tag{3.4}$$

Combining Lemma 3.1 and Lemma 3.4, we get another segment inequality lemma as follows.

Lemma 3.5. *Let $b^+(q) = d(q,q^+) - d(p,q^+)$ for any q with $d(q,q^+) \geq 10$. Let h^+ be a smooth function which satisfies*

$$\int_{B_p(1)} |\nabla h^+ - \nabla b^+| \leq \epsilon vol^f(B_p(1))$$

and

$$\int_{B_p(1)} |hess\ h^+| \leq \epsilon vol^f(B_p(1)).$$

We assume that Lemma 3.3 and Lemma 3.4 are true. Then for any two points $q, q' \in B_p(\frac{1}{8})$ and any small number $\eta > 0$, there exist y^*, z^* with $d(y^*,q) < \eta, d(z^*,q') < \eta$, and a minimal geodesic line $\gamma(t)$ $(0 \leq t \leq l(y^*))$ from y^* to q^{-1} with $\gamma(0) = y^*, \gamma(l(y^*)) \in \partial B_p(\frac{1}{8})$ such that the following is true:

$$\int_0^{l(y^*)} |\nabla h^+(s) - \gamma'(s)|ds \leq \epsilon\frac{vol^f(B_q(2))}{vol^f(B_q(\eta))}, \tag{3.5}$$

$$\int_0^{l(y^*)} ds\int_0^{d(z^*,\gamma(s))} |hess\ h^+(\gamma_s(t))|dt \leq \epsilon(\frac{vol^f(B_q(2))}{vol^f(B_q(\eta))})^2, \tag{3.6}$$

where $\gamma_s(t)$ is the minimal geodesic curvse connecting $\gamma(s)$ and q'.

Proof. Choose a cut-off function $\phi = \phi(\mathrm{dist}(p,\cdot))$ with support in $B_p(1)$. Let

$$e = \phi|\nabla h^+ - \nabla b^+|, e_1 = \phi|\text{hess } h^+|,$$

$$e_2(y) = \int_{B_{q'}(\eta)} dz \int_0^{d(y,z)} e_1(\gamma_{yz})(s)ds.$$

Then by Lemma 3.4, we have

$$\int_{B_q(\eta)} \int_0^{d(q^-,y)} e(\gamma_{q^-y}(s))dsdy \le c(A,\Lambda) \int_{B_p(1)} e(y)dy. \tag{3.7}$$

On the other hand, by Lemma 3.3 , one sees

$$\begin{aligned}\int_{B_p(1)} e_2(y)dy &= \int_{B_p(1)} dy \int_{B_{q'}(\eta)} dz \int_0^{d(y,z)} e_1(\gamma_{yz})(s)ds \\ &\le c_1(\Lambda, A)\mathrm{vol}^{\,f}(B_p(1)) \int_{B_p(1)} e_1(y)dy.\end{aligned}$$

Thus by Lemma 3.4, we get

$$\begin{aligned} &\int_{B_q(\eta)} \int_0^{d(q^-,y)} e_2(\gamma_{q^-y}(s))dsdy \\ &\le c_2(\Lambda, A) \int_{B_p(1)} e_2(y)dy \\ &\le \mathrm{vol}^{\,f}(B_p(1))c_3(\Lambda, A) \int_{B_p(1)} e_1(y)dy. \end{aligned} \tag{3.8}$$

Observe that the left hand side of (3.8) is equal to

$$\int_{B_q(\eta)} dy \int_{B_{q'}(\eta)} dz \int_0^{d(q^-,y)} \int_0^{d(\gamma_{q^-y}(s),z)} e_1(\hat{\gamma}_s(t))dtds,$$

where $\hat{\gamma}_s(t)$ is the minimal geodesic from z to $\gamma_{q^-y}(s)$ with arc-length parameter t. Combining (3.7) and (3.8), we find two points y^*, z^* such that both (3.5) and (3.6) are satisfied.

□

Now we apply Lemma 3.5 to prove a local version of Theorem 3.1.

Proposition 3.6. *Let (M,g) be a complete Riemannian manifold which satisfies*

$$Ric_M^f(g) \ge -\frac{n-1}{R^2}, \ |f|, \ |\nabla f| \le A.$$

Suppose that there exist three points p, q^+, q^- such that

$$d(p,q^+) + d(p,q^-) - d(q^+,q^-) < \epsilon \tag{3.9}$$

and

$$d(p, q^+) \geq R, d(p, q^-) > R. \tag{3.10}$$

Then there exists a map

$$u : B_p(1/8) \longrightarrow B_{(0,x)}(1/8) \tag{3.11}$$

as a $\Psi(1/R, \epsilon; A, n)$ *Gromov-Hausdorff approximation, where* $B_{(0,x)}(1/8) \subset \mathbb{R} \times X$ *is a* $\frac{1}{8}$*-radius ball centered at* $(0, x) \in \mathbb{R} \times X$ *and* X *is given by the level set* $(h^+)^{-1}(0)$ *as a metric space measured in the* $B_p(1)$*.*

Proof. For simplicity, we denote the terms on the right-hand side of (2.8), (2.9) and (2.10) in Lemma 2.3 by $\delta = \delta(\epsilon, \frac{1}{R})$. Define a map u on $B_p(1)$ by $u(q) = (x_q, h^+(q))$, where x_q is the nearest point to q in X. We are going to prove that u is a $\Psi(1/R, \epsilon; A)$ Gormov-Hausdorff approximation. Since $|\nabla h^+| \leq c = c(A)$ in $B_p(\frac{1}{2})$,

$$h^+(y) \leq 0, \ \forall \ y \in B_q(\eta), \ \text{if } h^+(q) < -c\eta,$$

where η is an appropriate small number and it will be determined late. We call the area of $h^+(q) < -c\eta$ the upper region, $h^+(q) > c\eta$ the lower region and the rest the middle region, respectively.

Case 1. Both points q_1 and q_2 in the upper region (we may assume that $h^+(q_1) > h^+(q_2)$). Let q be a point in the upper region. Then by applying Lemma 3.5 to q, x_q, we get a geodesic from a point y near q to q^- whose direction is almost the same as ∇h^+. Thus this geodesic must intersect $h^+ = 0$. Applying Triangular Lemma 3.2, we see that the intersection is near x_q. Hence for q_1 and q_2, we can find y_1 and y_2 nearby q_1 and q_2 respectively, such that two geodesics from y_1 and y_2 to q^- intersect X with points x_1 and x_2, respectively. Denote the geodesic from x_2 to y_2 by $\gamma(s) : \gamma(0) = x_2, \gamma(h^+(y_2)) = y_2$. Applying Triangular Lemma 3.2 to triples $\{y_1, y_2, \gamma(h^+(y_1))\}, \{x_2, y_1, \gamma(h^+(y_1))\}$ and $\{x_1, x_2, y_1\}$, respectively, we get

$$|d(y_1, y_2)^2 - |h^+(y_2) - h^+(y_1)|^2 - d(y_1, \gamma(h^+(y_1)))^2| \leq c(n, A)\frac{\delta}{\eta^n},$$

$$|d(y_1, x_2)^2 - d(y_1, \gamma(h^+(y_1)))^2 - h^+(y_1)^2| \leq c(n, A)\frac{\delta}{\eta^n},$$

and

$$|d(y_1, x_2)^2 - d(x_1, x_2)^2 - h^+(y_1)^2| \leq c(n, A)\frac{\delta}{\eta^n}.$$

Combining the above three relations, we derive

$$|d(q_1, q_2) - d(u(q_1), u(q_2))| \leq c(n, A)\frac{\delta}{\eta^n} << 1 \tag{3.12}$$

as $\delta = o(\eta^n)$.

Case 2. q_1 is in the middle region and q_2 is in the upper region. Note that x_q is near q if q is in the middle region. Then we can find two points y_1 and y_2 near q_1 and q_2 respectively, such that Triangular Lemma 3.2 holds for the triple $\{y_1, y_2, x_2\}$. Hence for such two points q_1 and q_2, we get (3.12) immediately.

Case 3. q_1 is in the lower region and q_2 is in the upper region. As in Case 1. we can get one geodesic from q^+ to a point near q_1 and another geodesic from q^- to a point near q_2, respectively. Thus we can use same argument in Case 1 to obtain (3.12). Similarly, we can settle down another two cases, both q_1 and q_2 in the lower region and both q_1 and q_2 in the middle region. $\square$

Proof of Theorem3.1. Suppose that the line in Y is $\gamma(t)$ and $\gamma(0) = y$. Define a Busemann function b along γ by

$$b(y) = \lim_{t\to+\infty}(d(y, \gamma(t)) - t).$$

Since

$$d_{GH}(B_{p_i}(j), B_y(j)) \to 0, \text{ as } i \to \infty,$$

for any given integer number $j > 0$, we may assume that

$$d_{GH}(B_{p_i}(j), B_y(j)) < \frac{1}{j}, \epsilon_i < \frac{n-1}{j^2} \text{ for } i = i(j) \text{ large enough.}$$

Choose a Gromov-Hausdorff approximation from $B_y(j)$ to $B_{p_i}(j)$ so that the images of endpoints $\gamma(j)$ and $\gamma(-j)$ of the line in $B_y(j)$ together with p_i satisfy the condition (3.10) in Proposition 3.6. Then we see that there exist a metric space X_j and a Gromov-Hausdorff approximation $u_j : B_{p_i}(1) \to B_{0\times x_j}(1)$ such that

$$d_{GH}(B_{p_i}(1), u_j(B_{p_i}(1))) < \Psi(\frac{1}{j}).$$

As a consequence, there exists a map $\hat{u}_j : B_y(1) \to B_{0\times x_j}(1)$ such that

$$d_{GH}(B_y(1), \hat{u}_j(B_y(1))) < \Psi.$$

This implies that all the projection of $\mathbb{R}$ component from space $\mathbb{R} \times X_j$ are close to the Buseman function b along the given line in Y for $j >> 1$, so they are almost the same. Hence, $\{X_j\}$ is a Cauchy sequence in Gromov-Hausdorff topology with a limit X. It follows that $B_y(1) = B_{0\times x}(1)$ where x is the limit point of $\{x_j\}$ in X. Since the number 1 can be replaced by any positive number, we finish the proof of theorem. $\square$

4. Existence of metric cone

In this section, we prove the existence of metric cone of a tangent space on a limit space of a sequence in $\mathcal{M}(A, v, \Lambda)$. Recall

Definition 4.1. *For a metric space (Y, d), the limit of $(Y, \epsilon_i^{-2}d; y)$ in Gromov Hausdorff topology as $\epsilon_i \to 0$ is called a tangent cone of Y at y (if exists). We denote it as T_yY.*

Definition 4.2. *Given a metric space X, the space $\mathbb{R}^+ \times X$ with the metric defined by*

$$d((r_1, x_1), (r_2, x_2)) = \sqrt{r_1^2 + r_1^2 - 2r_1r_1 \cos d(x_1, x_2)}, \text{ if } d(x_1, x_2) \le \pi,$$
$$d((r_1, x_1), (r_2, x_2)) = r_1 + r_2, \text{ if } d(x_1, x_2) \ge \pi$$

is called a metric cone over X. We usually denote it by $C(X)$ with the metric $\mathbb{R}^+ \times_r X$.

The main theorem of this section can be stated as follows.

Theorem 4.3. *Let $\{(M_i, g_i; p_i)\}$ be a sequence of manifolds in $\mathcal{M}(A, v, \Lambda)$. Then there exists a subsequence of $\{(M_i, g_i; p_i)\}$ converges to a metric space $(Y; y)$ in the pointed Gromov-Hausdorff topology. Moreover, for each $z \in (Y; y)$, each tangent cone T_zY is a metric cone over another metric space whose diameter is less than π.*

The proof of Theorem 4.3 is similar to one of Splitting Theorem 3.1. We need another Cheeger's triangluar lemma to estimate the distance.

Lemma 4.4. *(Lemma 9.64, [C1]) Let x, y be two ponis in a minimal geodesic curve from p and denote the part of the geodesic curve from x to y by $\gamma(s)$. Let $\gamma_s(t)$ be a family of geodesic curves connecting z and $\gamma(s)$ as in Lemma 3.2. Suppose that there is a smooth function h such that*

$$\int_{[0,a]} \int_{[0,l(s)]} |hess\ h - g| dtds < \delta << 1 \tag{4.1}$$

and

$$\int_{[0,a]} |\nabla h(\gamma(s)) - r(\gamma(s))\gamma'(s)| < \delta << 1, \tag{4.2}$$

where $r(q) = dist(p, q)$. Then

$$\begin{aligned} &d(z, y)^2 r(x) + d(z, p)^2 (r(y) - r(x)) \\ &- d(x, z)^2 r(y) - r(x)r(y)(r(y) - r(x)) < \epsilon(\delta). \end{aligned} \tag{4.3}$$

It is easy to see the left-hand side of (4.3) is zero in a metric cone $C(X)$ if x, y lie in a radial direction.

We need a few of lemmas to prove Theorem 4.3.

Lemma 4.5. *Given $\eta > 0$, there exists $\omega = \omega(a, b, \eta, A, \Lambda)$ such that the following is true: if*

$$Ric_M^f(g) \geq -(n-1)\Lambda^2 g \text{ and } |\nabla f| \leq A \tag{4.4}$$

and

$$\frac{vol^f(\partial B_p(b))}{vol^f(\partial B_p(a))} \geq (1-\omega)\frac{L_{\Lambda,A}(b)}{L_{\Lambda,A}(a)}, \tag{4.5}$$

then for any point q on $\partial B_p(a)$, there exists q' on $\partial B_p(b)$ such that

$$d(q, q') \leq b - a + \eta.$$

Proof. Suppose the conclusion fails to hold for some η and $q_1 \in \partial B_p(a)$. Then for any point in $B_{q_1}(\frac{\eta}{3})$, there is no point q on $\partial B_p(b)$ such that $d(q_1, q) \leq b - a + \frac{\eta}{3}$. Thus for any $r < \frac{\eta}{3}$, any minimal geodesic from p to $\partial B_p(b)$ does not intersect with $B_{q_1}(\frac{\eta}{3}) \cap \partial B_p(a+r)$. Since

$$\mathrm{vol}^f(B_{q_1}(\frac{\eta}{3})) \geq \frac{L_{\Lambda,A}(\frac{\eta}{3})}{L_{\Lambda,A}(2b)}\mathrm{vol}^f(A_p(a,b)),$$

by the coarea formula, there exists some $\frac{\eta}{4} < r < \frac{\eta}{3}$ such that

$$\mathrm{vol}^f(B_{q_1}(\frac{\eta}{3}) \cap \partial B_p(a+r)) \geq \frac{1}{\eta}\frac{L_{\Lambda,A}(\frac{\eta}{3})}{L_{\Lambda,A}(2b)}\mathrm{vol}^f(A_p(a,b)).$$

Using the monotonicity formula (1.9), we get

$$\begin{aligned}
\mathrm{vol}^f(\partial B_p(b)) &\leq \mathrm{vol}^f(\partial B_p(a+r) \setminus B_{q_1}(\frac{\eta}{3}))\frac{L_{\Lambda,A}(b)}{L_{\Lambda,A}(a+r)} \\
&\leq (\mathrm{vol}^f(\partial B_p(a+r)) - \frac{1}{\eta}\frac{L_{\Lambda,A}(\frac{\eta}{3})}{L_{\Lambda,A}(2b)}\mathrm{vol}^f(A_p(a,b)))\frac{L_{\Lambda,A}(b)}{L_{\Lambda,A}(a+r)}.
\end{aligned}$$

It follows

$$\begin{aligned}
\mathrm{vol}^f(\partial B_p(b)) &\leq (1+\delta'(\eta,b,a))^{-1}\mathrm{vol}^f(\partial B_p(a+r))\frac{L_{\Lambda,A}(b)}{L_{\Lambda,A}(a+r)} \\
&\leq (1+\delta'(\eta,b,a))^{-1}\mathrm{vol}^f(\partial B_p(a))\frac{L_{\Lambda,A}(b)}{L_{\Lambda,A}(a)}.
\end{aligned}$$

But this is a contradiction to (4.5) as $\omega < \frac{1}{2}\delta'(\eta, b, a)$. Therefore, the lemma is proved. □

By applying Theorem 3.6 in $[CC1]$ with help of Lemma 4.4 and Lemma 4.5, we have the following proposition.

Proposition 4.6. *Given $\eta > 0$, there exist $\omega = \omega(a, b, \eta)$ and $\delta = \delta(\eta)$ such that if (4.4) and (4.5) are satisfied, then there is a length space X such that*

$$d_{GH}(A_p(a,b), (a,b) \times_r X) < \eta,$$

where $(a,b)\times_r X$ is an annulus in $C(X)$ and the metric of $A_p(a,b)$ is measured in a slightly bigger annulus in M.

Proof. It suffices to verify the distance function condition in Theorem 3.6 in $[CC1]$. Let x,y,z,w be four points in the annulus $A_p(a,b)$ such that both pairs $\{x,y\}$ and $\{z,w\}$ are in the radial direction from p. Then by applying the segment inequality of Lemma 3.5 to the function h in Lemma 2.4 and Lemma 2.5, we can find another four points x_1,y_1,z_1,w_1 near the four points respectively such that Triangular Lemma 4.4 holds for two triples $\{x_1,y_1,z_1\}$ and $\{y_1,z_1,w_1\}$. Now we choose four points x_2,y_2,z_2,w_2 in the plane $\mathbb{R}^2$ such that both triples $\{O,x_2,y_2\}$ and $\{O,z_2,w_2\}$ are co-linear. Moreover, we can require that

$$r(x_2)=r(x_1),r(y_2)=r(y_1),r(z_2)=r(z_1),r(w_2)=r(w_1)$$

and

$$d(x_1,z_1)=d(x_2,z_2).$$

Thus by using Triangular Lemma 4.4 to $\{x_1,y_1,z_1\}$, it is easy to see that

$$|d(y_2,z_2)-d(y_1,z_1)|<\Psi. \tag{4.6}$$

Applying Triangular Lemma 4.4 to $\{y_1,z_1,w_1\}$, we have

$$\begin{aligned}|d(y_1,z_1)^2r(w_1)+r(w_1)r(z_1)(r(w_1)-r(z_1))\\ -d(y_1,w_1)^2r(z_1)-r(y_1)^2d(z_1,w_1)|<\Psi.\end{aligned} \tag{4.7}$$

Note that the left hand side of (4.7) is zero when the triple $\{y_1,z_1,w_1\}$ is replaced by $\{y_2,z_2,w_2\}$ in the plane. Since

$$|d(z_1,w_1)-(r(w_1)-r(z_1))|<\Psi,$$

we get from (4.6) and (4.7) that,

$$|d(y_1,w_1)-d(y_2,w_2)|<\Psi.$$

On the other hand, $d(y_2,w_2)$ can be written as the following function:

$$d(y_2,w_2)=Q(r(x_2),r(y_2),r(z_2),r(w_2),d(x_2,z_2)).$$

Therefore

$$|d(y_1,w_1)-Q(r(x_1),r(y_1),r(z_1),r(w_1),d(x_1,z_1))|<\Psi.$$

It follows that

$$|d(y,w)-Q(r(x),r(y),r(z),r(w),d(x,z))|<\Psi. \tag{4.8}$$

(4.8) is just the condition for distance function in Theorem 3.6 in [CC1].

By (4.8) and Lemma 4.5 we see that two conditions in Theorem 3.6 in [CC1] are satisfied. Hence as a consequence of this theorem, we obtain

Proposition 4.6. In fact, X is a level set of $r^{-1}(a)$ with a χ-intrinsic metric defined by

$$l^{\chi}(x,y) = \frac{1}{a}\inf \Sigma_{i=1}^{n} d(x_{i-1},x_i), \tag{4.9}$$

where the infimum is taken among all the sequences $\{x_i\} \in X$ which satisfy $x_0 = x, x_n = y$ and $d(x_{i-1},x_i) \le \chi$.

□

It remains to verify the condition (4.5) in Lemma 4.5.

Lemma 4.7. *Given $0 < a < b = a\Omega, \Omega > 0$, there exists an integer $N = N(n,\Omega,\Lambda,v,A)$ such that for any sequence of r_i $(1 \le i \le N)$ with $\Omega r_{i+1} \le r_i \le \frac{1}{b}$, the volume condition (4.5) for any manifold $(M,g) \in \mathcal{M}(\Lambda,v,A)$ in Lemma 4.5 holds for some annulus $A_p(ar_k,br_k) \subset M$ $(1 \le k \le N)$ with rescaling metric $\hat{g} = \frac{g}{r_k}$.*

Proof. We only need to give an upper bound of N in case that the following inequality

$$\frac{\mathrm{vol}^f_{\hat{g}}(\partial B_p(br_k))}{L_{r_k\Lambda,r_kA}(br_k)} \ge e^{-\omega}\frac{\mathrm{vol}^f_{\hat{g}}(\partial B_p(ar_k))}{L_{r_k\Lambda,r_kA}(ar_k)} \tag{4.10}$$

doesn't hold for any $1 \le k \le N$. Then by the monotonicity formula (1.9), we know that

$$\frac{\mathrm{vol}^f_{\hat{g}}(\partial B_p(br_N))}{L_{r_k\Lambda,r_kA}(br_N)} \le e^{-N\omega}\frac{\mathrm{vol}^f_{\hat{g}}(\partial B_p(br_1))}{L_{r_k\Lambda,r_kA}(br_1)}.$$

Thus by the non-collapsing condition the left-hand side has a lower bound $c_1(n,\Lambda,v,A)$, and by Volume Comparison Theorem 1.2 the right-hand side is not greater than $e^{-N\omega}c_2(n,\Lambda,v,A)$. Thus this helps us to get an upper bound of N. Hence, if N is larger than this bound, there must be some k such that (4.10) holds. The lemma is proved. □

Proof of Theorem4.3. Without loss of generality, we may assume that $z = y$ since each point in $(Y,d;y)$ is a limit of sequence of volume non-collapsing points in M_i. Also we note that the tangent cone T_yY always exists in our case by Gromov's theorem [Gr]. By the contradiction argument, we suppose that T_yY is not a metric cone. Then it is easy to see that there exist numbers $0 < a < b, \eta_0 > 0$ and a sequence $\{r_i\}$, which tends to 0, such that for any length space X annulus $A_y(ar_i,br_i) \subset (Y,\frac{d}{r_i};y)$ satisfy,

$$d_{GH}(A_y(ar_i,br_i),(ar_i,br_i)\times_r X) > 3r_i\eta_0. \tag{4.11}$$

By taking a subsequence we may assume that $\Omega r_{i+1} \le r_i$ $(\Omega = \frac{b}{a})$ and r_i is smaller than δ in Lemma 4.5. On the other hand, since Y is the limit of

M_i, we can find an increasing sequence m_i such that for every $j \geq m_i$

$$d_{GH}(A_y(ar_i, br_i), A_{p_j}(ar_i, br_i)) < r_i\eta_0. \tag{4.12}$$

Let ω be a small number as chosen in Proposition 4.6 and N an integer such that Lemma 4.7 is true for the $\omega > 0$. Thus by (4.12), we see that there exist a subsequence $\{r_{i_k}\} \to 0$ and a sequence $\{j_k\} \to \infty$ such that

$$d_{GH}(A_y(ar_{i_k}, br_{i_k}), A_{p_{j_k}}(ar_{i_k}, br_{i_k})) < r_{i_k}\eta_0, \tag{4.13}$$

where annulus $A_{p_{j_k}}(ar_{i_k}, br_{i_k})$ are chosen as in Lemma 4.7. Now we can apply Proposition 4.6 to show that for each large k there exists a length space X such that

$$d_{GH}(A_{p_{j_k}}(ar_{i_k}, br_{i_k}), (ar_{i_k}, br_{i_k}) \times_r X) < r_{i_k}\eta_0.$$

But this is impossible by (4.11). Therefore, T_yY must be a metric cone.

The diameter estimate follows from Splitting Theorem 3.1. In fact, if $diam(X) > \pi$, there will be two points p, q in X such that $d(p, q) = \pi$. By Theorem 3.1, it follows that $C(X) = \mathbb{R} \times Y_1$, where Y_1 is also a metric cone, i.e. $Y_1 = C(X_1)$. It is clear that $diam(X_1) > \pi$ since $diam(X) > \pi$. Thus we can continue to apply Theorem 3.1 to split off X_1. By the induction, $C(X)$ should be an Euclidean space, and consequently X is a standard sphere . But this is impossible by the assumption that $diam(X) > \pi$.

$\square$

Following the argument in the proofs of Theorem 4.3 and Proposition 4.6, we actually prove the following strong approximation of Gromov-Hausdorff to the flat space.

Corollary 4.8. *For all $\epsilon > 0$, there exists $\delta = \delta(n, \epsilon), \eta = \eta(n, \epsilon)$ such that if*

$$Ric_M^f(g) \geq -(n-1)\delta^2 g, \ |\nabla f| \leq \eta \tag{4.14}$$

and

$$e^{-f(0)} vol^f(B_p(1)) \geq (1-\delta) vol(B_0(1)) \tag{4.15}$$

are satisfied, then

$$d_{GH}(B_p(1), B_0(1)) < \epsilon. \tag{4.16}$$

Proof. Suppose that the conclusion (4.16) is not true. Then there exist sequences of $\{\delta_i\}$ and $\{\eta_i\}$ which tend 0 both, and a sequence of manifolds $\{(M, g_i)\}$ with conditions (4.14) and (4.15) such that

$$d_{GH}(B_{p_i}(1), B_0(1)) \geq \epsilon_0 > 0, \tag{4.17}$$

where $B_{p_i}(1) \subset M_i$. Then following the argument in the proofs of Theorem 4.3 and Proposition 4.6, it is no hard to show that $B_{p_i}(1)$ converge to a limit

$B_x(1)$ which is a metric ball with radius 1 in a metric-cone $(C(X), d)$ with vertex x. Since the blowing-up space of $B_x(1)$ at x is $C(X)$ itself, we see that there are subsequences $\{j\}$ and $\{i_j\}$, both of which tend to infinity, such that

$$(B_{p_{i_j}}(j), j^2 g_{i_j}, q_{i_j}) \to (C(X), d, x).$$

For any $y \in X$, we choose a sequence of points $q_{i_j} \in B_{p_{i_j}}(j) \subset (M_{i_j}, j^2 g_{i_j})$ which tends to y. Then for any given $R > 0$, we have

$$B_{q_{i_j}}(R)(\subseteq (M_{i_j}, j^2 g_{i_j})) \to B_y(R).$$

Since the volume condition (4.15) implies

$$e^{-f(0)} \mathrm{vol}_f(B_{q_{i_j}}(R)) \to \mathrm{vol}(B_0(R)), \tag{4.18}$$

by the above argument, $B_y(R)$ is in fact a metric ball with radius R in a metric cone $C(Y)$ with vertex y. Note that R is arbitrary. We prove that $C(X)$ is also a cone with vertex at y. This shows that there exists a line connecting x and y in $C(X)$. By Splitting Theorem 3.1, $C(X)$ can split off a line along the direction xy. Since $y \in X$ can be taken in any direction, $C(X)$ must be an euclidean space. But this is impossible according to (4.17). The Corollary is proved.

□

Remark 4.9. *Corollary 4.8 is a generalization of Theorem 9.69 in [C1] in Bakry-Émery geometry. It will be used in Section 5 and Section 6 for the blowing-up analysis. We also note that $e^{-f(0)} vol^f(B_p(1))$ is close to $vol(B_p(1))$ since $|\nabla f|$ is small enough. Thus the volume condition (4.15) can be replaced by*

$$vol(B_p(1)) \geq (1-\delta) vol(B_0(1)).$$

For the rest of this section, we prove the Colding's volume convergence theorem in Bakry-Émery geometry by using Hessian estimates in Section 2.

Theorem 4.10. *Let (M_i^n, g_i) be a sequence of Riemannian manifolds which satisfy (4.4). Suppose that M_i converge to an n-dimensional compact manifold M in Gomov-Hausdorff topology. Then*

$$\lim_{i\to\infty} vol(M_i, g_i) = vol(M).$$

We first prove a local version of Theorem 4.10 as follows.

Lemma 4.11. *Given $\epsilon > 0$, there exist $R = R(\epsilon, \Lambda, A, n) > 1$ and $\delta = \delta(\epsilon, \Lambda, A, n)$ such that if*

$$Ric_M^f(g) \geq -(n-1)\frac{\Lambda^2}{R^2} g, |\nabla f| \leq \frac{A}{R}, \tag{4.19}$$

and

$$d_{GH}(B_p(R), B_0(R)) < \delta, \tag{4.20}$$

then we have

$$vol(B_p(1)) > vol(B_0(1)) - \epsilon. \tag{4.21}$$

Proof. We need to construct a Gromov-Hausdorff approximation map by using f-harmonic functions constructed in Section 2. Choose n points q_i in $B_p(R)$ which is close to Re_i in $B_0(R)$, respectively. Let $l_i(q) = d(q, q_i) - d(q_i, p)$ and h_i a solution of

$$\Delta^f h_i = 0, \text{ in } B_1(p),$$

with $h_i = l_i$ on $\partial B_1(p)$. Then by Lemma 2.3, we have

$$\frac{1}{\text{vol }(B_p(1))}\int_{B_p(1)} |\text{hess } h_i|^2 < \Psi(1/R, \delta; A).$$

By using an argument in [Co] (cf. Lemma 2.9), it follows

$$\frac{1}{\text{vol }(B_p(1))}\int_{B_p(1)} |\langle \nabla h_i, \nabla h_j\rangle - \delta_{ij}| < \Psi(1/R, \delta; A). \tag{4.22}$$

Define a map by $h = (h_1, h_2, ..., h_n)$. It is easy to see that the map h is a $\Psi(\frac{1}{R}, \delta; \Lambda)$ Gromov-Hausdorff approximation to $B_p(1)$ by using the estimate (2.8) in Lemma 2.3. Since h maps $\partial B_p(1)$ nearby $\partial B_0(1)$ with distance less than Ψ, by a small modification to h we may assume that

$$h : (B_p(1), \partial B_p(1)) \longrightarrow (B_0(1-\Psi), \partial B_0(1-\Psi)).$$

Next we us a degree argument in [C2] to show that the image of h contains $B_0(1-\Psi)$. By using Vitali covering lemma, there exists a point x in $B_p(\frac{1}{8})$ such that for any r less than $\frac{1}{8}$ it holds

$$\frac{1}{\text{vol }(B_x(r))}\int_{B_x(r)} |\text{hess } h_i| < \Psi \tag{4.23}$$

and

$$\frac{1}{\text{vol }(B_x(r))}\int_{B_x(r)} |\langle \nabla h_i, \nabla h_j\rangle - \delta_{ij}| < \Psi. \tag{4.24}$$

Let $\eta = \Psi^{\frac{1}{2n+1}}$. For any y with $d(x, y) = r < \frac{1}{8}$, applying Lemma 3.3 to $A_1 = B_x(\eta r), A_2 = B_y(\eta r), e = |\text{hess } h_i|$, we get from (4.23),

$$\begin{aligned}&\int_{B_x(\eta r)\times B_y(\eta r)}\int_{\gamma_{zw}} |\text{hess } h_i(\gamma', \gamma')|\\ &< r(\text{vol }(B_x(\eta r)) + \text{vol }(B_y(\eta r)))\text{vol }(B_x(r))\Psi.\end{aligned}$$

It follows that

$$\int_{B_x(\eta r)}[Q(r,\eta)\int_{B_y(\eta r)}\int_{\gamma_{zw}}\Sigma_{i=1}^n|\text{hess } h_i(\gamma',\gamma')|+|\langle\nabla h_i,\nabla h_j\rangle-\delta_{ij}|]$$
$$< \text{vol } (B_x(\eta r))\Psi,$$

where $Q(r,\eta)=\frac{\text{vol}B_x(\eta r)}{r(\text{vol }(B_x(\eta r))+\text{vol }(B_y(\eta r)))\text{vol}B_x(r)}$. Consider

$$Q(r,\eta)\int_{B_y(\eta r)}\int_{\gamma_{zw}}\Sigma_{i=1}^n\text{hess } |h_i(\gamma',\gamma')|+|\langle\nabla h_i,\nabla h_j\rangle-\delta_{ij}|$$

as a function of $z\in B_x(\eta r)$. Then one sees that there exists a point x^* such that

$$|\langle\nabla h_i,\nabla h_j\rangle(x^*)-\delta_{ij}|<\Psi \tag{4.25}$$

and

$$\Sigma_{i=1}^n\int_{B_y(\eta r)}\int_{\gamma_{x^*w}}|\text{hess } h_i(\gamma',\gamma')|<r\text{vol }(B_x(r))\eta^{-n}\Psi. \tag{4.26}$$

Here at the last inequality, we used the volume comparison (1.5). Moreover by (4.26), we can find a point y^* such that

$$\Sigma_{i=1}^n\int_{\gamma_{x^*y^*}}|\text{hess } h_i(\gamma',\gamma')|<\eta r. \tag{4.27}$$

By a direct calculation with help of (4.25) and (4.27), we get

$$(h(x^*)-h(y^*))^2=(1+\Psi^{\frac{1}{2n+1}})r^2. \tag{4.28}$$

This shows that $h(x)\neq h(y)$ for any y with $d(y,x)\leq\frac{1}{8}$. On the other hand, for any y with $d(y,x)\geq\frac{1}{8}$, it is clear that $h(x)\neq h(y)$ since h is a Ψ Gromov-Hausdorff approximation. Thus we prove that the pre-image of $h(x)$ is unique. Therefore the degree of h is 1, and consequently, $B_0(1-\Psi)\subset h(B_p(1))$. The lemma is proved because the volume of $B_p(1)$ is almost same to one of $h(B_p(1))$ by the fact (4.22). □

Proof of Theorem 4.10. Choose finite balls $B(q_i,r_i)$ to cover M with r_i small enough to make all balls close to Euclidean balls so that

$$\Sigma_i\text{vol}(B(q_i,r_i))<(1+\epsilon)\text{vol}(M)$$

for any given $\epsilon>0$. Then for j sufficiently large, M_j can be covered by $B(q_{ji},r_{ji})$ with $r_{ji}\leq(1+\epsilon)r_i$. Thus by the volume comparison (1.5), we have

$$\begin{aligned}\text{vol}(M_j)&\leq\Sigma_i\text{vol}(B(q_{ji},r_{ji}))\\&<(1+\Psi(\delta:\Lambda,A))\Sigma_i\text{vol}(B(q_i,r_i)).\end{aligned} \tag{4.29}$$

Here $\delta = max\{r_i\}$. Hence we get

$$lim_{j\to\infty} Vol(M_j) \leq \mathrm{vol}(M).$$

On the other hand, for any $\epsilon > 0$, we choose small enough N disjoint balls $B(q_i, r_i)$ in M with $B(q_i, r_i)$ close to Euclidean balls so that

$$(1+\epsilon)\Sigma_i \omega_n r_i^n \geq \Sigma_i \mathrm{vol}(B(q_i, r_i)) > (1-\epsilon)\mathrm{vol}(M). \tag{4.30}$$

Then for a fixed large number R, we see that for j large enough there are corresponding disjoint balls $B(q_{ij}, r_i)$ in M_j such that $B(q_{ij}, Rr_i)$ is $\delta(N)$-close to $B(q_i, Rr_i)$ in Gromov-Hausdorff topology, where $\delta(N)$ is the number determined in Lemma 4.11 when ϵ is replaced by $\frac{\epsilon}{N}$. Apply the above lemma to each ball $B(q_{ij}, Rr_i)$ with rescaling metric $\frac{g_j}{r_i}$, we get from (4.30),

$$(1+\epsilon)\mathrm{vol}(M_j) > (1-\epsilon)\mathrm{vol}(M) - (1+\epsilon)\epsilon.$$

Taking ϵ to 0 and N to ∞, it follows

$$lim_{j\to\infty} Vol(M_j) \geq \mathrm{vol}(M).$$

The theorem is proved.

□

5. Structure of singular set I: Case of Riemannian metrics

According to Theorem 4.3, we may introduce a notion of S_k-typed singular point y in a limit space $(Y, d_\infty; p_\infty)$ of a sequence of Riemannian manifolds $\{(M_i, g_i; p_i)\}$ in $\mathcal{M}(A, v, \Lambda)$ as Definition 0.2, if there exists a tangent cone at y which can be split out an euclidean space $\mathbb{R}^k$ isometrically with dimension at most k. By applying Metric Cone Theorem 4.3 to appropriate tangent cone space T_y, we can follow the argument in [CC2] to show that dimension of S_k is less than k. Moreover, $\mathcal{S} = \mathcal{S}_{n-2}$, where $\mathcal{S} = \cup_{i=0}^{n-1} \mathcal{S}_i$. The latter is equivalent to that any tangent cone can't be the upper half space, which can be proved by using a topological argument as in the case of Ricci curvature bounded below (cf. Theorem 6.2 in [CC2]). Thus we have

Theorem 5.1. *Let $\{(M_i, g_i; p_i)\}$ be a sequence of Riemannian manifolds in $\mathcal{M}(A, v, \Lambda)$ and let $(Y, d_\infty; p_\infty)$ be its limit in the Gromov-Hausdorff topology. Then dim $\mathcal{S}_k \leq k$ and $\mathcal{S}(Y) = \mathcal{S}_{n-2}$.*

Remark 5.2. *By Theorem* 5.1*, one sees that $\mathcal{H}^n(\mathcal{S}) = 0$. Thus by Theorem 4.10, we have*

$$\lim_{i\to\infty} vol(M_i) = \mathcal{H}^n(Y), \tag{5.1}$$

if Y is a Gromov-Hausdorff limit of a sequence $\{(M_i^n, g_i)\}$ in $\mathcal{M}(A, v, \Lambda)$. Moreover, if $B_i(r) \subset M_i$ converge to $B_\infty(r) \subset Y$,

$$\lim_{i\to\infty} vol(B_i(r)) = \mathcal{H}^n(B_\infty(r)), \tag{5.2}$$

where $B_i(r)$ and $B_\infty(r)$ are radius r-balls in M_i and Y, respectively.

We define ϵ-regular points in Y.

Definition 5.3. *$y \in (Y; p_\infty)$ is called an ϵ-regular point if there exist an ϵ and a sequence $\{r_i\}$ such that*

$$dist_{GH}((B_y(1), \frac{1}{r_i} d_\infty), B_0(1)) < \epsilon, \text{ as } i \to \infty.$$

Here $B_0(1)$ is the unit ball in $\mathbb{R}^n$. We denote the set of those points by $\mathcal{R}_\epsilon$.

In this section, our main purpose is to generalize Theorem 1.15 in [CCT] to analyze the singular structure $\mathcal{S}$ under an additional assumption of L^p-integral bounded curvature.

Theorem 5.4. *Let $\{(M_i, g_i; p_i)\}$ be a sequence in $\mathcal{M}(A, v, \Lambda)$ and $(Y; p_\infty)$ its limit as in Theorem 5.1. Suppose that*

$$\frac{1}{vol(B_{p_i}(2))} \int_{B_{p_i}(2)} |Rm|^p < C. \tag{5.3}$$

Then for any $\epsilon > 0$, the following is true: i)

$$\mathcal{H}^{n-2p}(B_{p_\infty}(1) \setminus \mathcal{R}_{2\epsilon}) < \infty, \text{ if } 1 \le p < 2; \tag{5.4}$$

ii)

$$dim(B_{p_\infty}(1) \setminus \mathcal{R}_{2\epsilon}) \le n - 4, \quad \text{if } p = 2. \tag{5.5}$$

The theorem is a consequence of following result of ϵ-regularity.

Proposition 5.5. *For any $v, \epsilon > 0$, there exist three small numbers $\delta = \delta(v, \epsilon, n)$, $\eta = \eta(v, \epsilon, n)$, $\tau = \tau(v, \epsilon, n)$ and a big number $l = l(v, \epsilon, n)$ such that if (M^n, g) satisfies*

$$Ric_M^f(g) > -(n-1)\tau^2, |\nabla f| < \tau, vol\ (B_p(1)) \ge v, \tag{5.6}$$

$$\frac{1}{vol(B_p(3))} \int_{B_p(3)} |Rm| < \delta, \tag{5.7}$$

and for some metric space X,

$$d_{GH}(B_p(l), B_{(0,x)}(l)) < \eta \tag{5.8}$$

holds for $k = 2$ or 3, where $(0, x)$ is the vertex in $\mathbb{R}^{n-k} \times C(X)$, then

$$d_{GH}(B_p(1), B_0(1)) < \epsilon. \tag{5.9}$$

To prove Proposition 5.5, it suffices to prove that $\mathrm{vol}(B_p(1))$ is close to $\mathrm{vol}(B_0(1))$ according to Corollary 4.8. The latter is equivalent to show that $\mathrm{vol}(B_0(1))$ is close to $\mathrm{vol}(B_{o,x}(1))$ by Remark 5.2. Thus we shall estimate the volume of section X. In the following, we will use the idea in [CCT] to turn into estimating volume of a pre-image of X by constructing a Gromov-Hausdorff approximation.

Let h_i $(i = 1, ..., n-k)$ be $(n-k)$ f-harmonic functions on $B_p(3)$ with appropriate boundary values as constructed in the proof of Splitting Theorem 3.1 (cf. Proposition 3.6) and h an approximation of $\frac{r^2}{2}$ as constructed in the proof of Metric Cone Theorem 4.3 (cf. Lemma 2.4, Lemma 2.5), which is a solution of

$$\Delta^f h = n.$$

Let

$$w_0 = h^2 - \Sigma h_j^2.$$

Define w to be a solution of

$$\Delta^f w = 2k,\ w|\partial B_p(4) = w_0.$$

Then w is almost positive, so it can be transformed to be positive by adding a small number. Set

$$\mathbf{u}^2 = w + \Psi > 0.$$

We recall some estimates for functions h_i, h and w:

$$\frac{1}{\mathrm{vol}(B_p(3))}\int_{B_p(3)}(\Sigma_i|\mathrm{hess}\ h_i|^2 + \Sigma_{i\neq j}|\langle\nabla h_i, \nabla h_j\rangle| + \frac{1}{\mathrm{vol}(B_p(3))}\int_{B_p(3)}\Sigma_i(|\nabla h_i| - 1)^2 < \Psi, \tag{5.10}$$

$$\frac{1}{\mathrm{vol}(B_p(3))}\int_{B_p(3)}(|\nabla h - \nabla r|^2 + |\mathrm{hess}\ h - g|^2) < \Psi, \tag{5.11}$$

$$\frac{1}{\mathrm{vol}B_p(3)}\int_{B_p(3)}|\mathrm{hess}\ w_0 - \mathrm{hess}\ w|^2 < \Psi, \tag{5.12}$$

and

$$\frac{1}{\mathrm{vol}B_p(3)}\int_{B_p(3)}|\nabla w_0 - \nabla w|^2 < \Psi. \tag{5.13}$$

The first two estimates are proved in Section 2 (cf. Lemma 2.3, Lemma 2.4, Lemma 2.5). We note that condition (2.13) in both Lemma 2.4 and Lemma 2.5 is satisfied according to (5.2) in Remark 5.2. The others can also be obtained in a similar way.

We define maps Φ and Γ respectively by

$$\Phi = (h_j) : B_p(4) \longrightarrow \mathbb{R}^{n-k}$$

and

$$\Gamma = (h_j, \mathbf{u}) : B_p(4) \longrightarrow \mathbb{R}^{n-k+1}.$$

Let

$$V_{\Phi,u}(z) = \mathrm{vol}(\Phi^{-1}(z) \cap U_u),$$

where $U_u = \Gamma^{-1}(B_0^{n-k}(1) \times [0,u])$ for $u \leq 2$. Then

Lemma 5.6.

$$\frac{1}{vol(B_0^{n-k}(1))} \int_{B_0^{n-k}(1)} |V_{\Phi,u}(z) - \frac{u^k}{k} vol(X)| < \Psi. \tag{5.14}$$

Proof. Set

$$v_\Phi = \nabla h_1 \wedge ... \wedge \nabla h_{n-k}.$$

Then v_Φ is the Jacobian of Φ in $B_0^{n-k}(1)$. By (5.10), one can show that it is almost 1 almost everywhere in $B_0^{n-k}(1)$. In fact, the proof is the same to one of (4.22). Hence by the coarea formula, we get

$$\begin{aligned} &\frac{1}{\mathrm{vol}(B_0^{n-k}(1))} \int_{B_0^{n-k}(1)} V_{\Phi,u}(z) \\ &= \frac{1}{\mathrm{vol}(B_0^{n-k}(1))} \int_{U_u} |v_\Phi| = \frac{\mathrm{vol}(U_u)}{\mathrm{vol}(B_0^{n-k}(1))} + \Psi. \end{aligned} \tag{5.15}$$

To compute the variation of $V_{\Phi,u}(z)$, we modify $V_{\Phi,u}(z)$ to

$$J_{\Phi,u,\delta} = \int_{\Phi^{-1}(z)} \chi_\epsilon(|v_\Phi|^2)\psi_{u,\delta},$$

where $\psi_{\delta,u} = \xi(\mathbf{u}^2)$ with a cut-off function ξ which satisfies

$$\xi(t) = 1, \text{ for } t \in [0, ((1-2\delta)u)^2],$$

$$\xi(t) = 0 \text{ for } t \in [((1-\delta)u)^2, u^2],$$

and $\chi_\epsilon(t)$ is another cut-off function which satisfies

$$\chi_\epsilon(t) = 0, \text{ for } t \in [0, \epsilon],$$

$$\chi_\epsilon(t) = (1-\epsilon)t, \text{ for } t \in [2\epsilon, 1-\epsilon],$$

$$\chi_\epsilon(t) = 1, \text{ for } t \geq 1,$$

$$|\chi'_\epsilon(t)| \leq 3.$$

A direct computation shows that

$$\begin{aligned}\frac{\partial J_{\Phi,u,\delta}}{\partial z_j} &= \int_{\Phi^{-1}(z)\cap U_u} \chi'_\epsilon(|v_\Phi|^2)\sum_i a_{i,j}\nabla h_i(|v_\Phi|^2)\psi_{u,\delta}, \\ &+ \int_{\Phi^{-1}(z)\cap U_u} \chi_\epsilon(|v_\Phi|^2)\sum_i a_{i,j} tr(\widehat{\text{hess } h_i})\psi_{u,\delta}, \\ &+ \int_{\Phi^{-1}(z)\cap U_u} \chi_\epsilon(|v_\Phi|^2)\sum_i a_{i,j}\langle\nabla\psi_\delta, \nabla h_i\rangle. \end{aligned} \tag{5.16}$$

Here $a_{i,j}$ is the inverse of $\langle\nabla h_i, \nabla h_j\rangle$ so that $\Phi_*(\sum_i a_{i,j}\nabla h_i) = \frac{\partial}{\partial z_j}$, and $tr(\widehat{\text{Hess } h_i})$ denotes the trace restricted to $\Phi^{-1}(z)$. Using the coarea formula the integrations of the first two terms at the right side of (5.16) in $B_0^{n-k}(1)$ can be controlled by the Hessian estimate in (5.10). Moreover, similar to (4.22), by (5.10) and (5.12), one can show,

$$\frac{1}{\text{vol}(B_p(1))}\int_{B_p(1)} |\langle\nabla \mathbf{u}^2, \nabla h_j\rangle| < \Psi.$$

Thus the integration of the third term at the right side of (5.16) in $B^{n-k}(1)$ is also small. Hence we get

$$\frac{1}{\text{vol}(B_0^{n-k}(1))}\int_{B_0^{n-k}(1)} |\nabla J_{\Phi,u,\delta}| < \Psi.$$

On the other hand, by (5.2), it is easy to see

$$\left|\frac{\text{vol}(U_u)}{\text{vol}(B_0^{n-k}(1))} - \frac{u^k}{k}\text{vol}(X)\right| < \Psi.$$

Therefore, we derive (5.14) from (5.15), ∎

Similar to (5.14), by using the above argument to the map Γ, one can also obtain the following estimate,

$$\frac{1}{\text{vol}(B^{n-k}(1)) \times [0,1])}\int_{B^{n-k}(1))\times[0,1]} |V_\Gamma(z,u) - u^{k-1}\text{vol}(X)| < \Psi, \tag{5.17}$$

where $V_\Gamma(z,u) = \text{vol}(\Gamma^{-1}(z,u))$. A similar proof can be also found in Theorem 2.63 in [CCT], so we omit it. Thus we see

Lemma 5.7. *There exists a subset of $D_{\epsilon,l} \subseteq B^{n-k}(1)\times[0,1]$ which depending only on ϵ, l such that*

$$vol(D_{\epsilon,l}) > (1-\Psi)vol(B^{n-k}(1)\times[0,1]) \tag{5.18}$$

and

$$|V_\Gamma(z,u) - u^{k-1}vol(X)| < \Psi, \ \forall\ (z,u)\in D_{\epsilon,l}. \tag{5.19}$$

Next, we use the Bochner identity in terms of Bakry-Emery Ricci curvature to estimate the second fundamental forms of pre-image of Φ, Γ. Let $v_1, v_2, ..., v_m$ be m smooth vector fields. Put $v = v_1 \wedge v_2 \wedge ... \wedge v_m$. We compute

$$\Delta^f |v|^2 = 2\langle \Delta^f v, v\rangle + 2|\nabla v|^2$$

and

$$\begin{aligned}\Delta^f (|v|^2+\eta)^{\frac{1}{2}} &= (|v|^2+\eta)^{-\frac{1}{2}}(|\nabla v|^2 - \frac{\langle \nabla v, v\rangle^2}{|v|^2+\eta})\\ &+ (|v|^2+\eta)^{-\frac{1}{2}}\langle \Delta^f v, v\rangle,\ \forall\ \eta>0.\end{aligned}$$

It follows,

$$\begin{aligned}&(|v|^2+\eta)^{-\frac{1}{2}}|\pi(\nabla v)|^2\\ &\leq -\frac{|v|}{(|v|^2+\eta)^{\frac{1}{2}}}(I-\pi)\Delta^f v + \Delta^f(|v|^2+\eta)^{\frac{1}{2}},\end{aligned} \tag{5.20}$$

where $\pi : \wedge^m \mathrm{TM} \to v^{\perp}$ is the compliment of orthogonal projection to v. On the other hand, if we choose $v_i = \nabla l_i$ and take map $F = (l_1, ...l_m)$ and $v = v_F$, then

$$|v_F||\Pi_{F^{-1}(c)}|^2 \leq |v_F|^{-1}|\pi(\nabla v_F)|^2, \tag{5.21}$$

where $\Pi_{F^{-1}(c)}$ denote the second fundamental form of the level set $F^{-1}(c)$ in M. Hence the quantity $(I-\pi)\Delta^f v_F$ in (5.20) gives us an estimate for the second fundamental form of map F.

To estimate $(I-\pi)\Delta^f v_F$, we use the following formula,

$$\Delta^f \nabla l_i = \nabla \Delta^f l_i + \mathrm{Ric}^f(\nabla l_i, \cdot).$$

Note that in our case $\Delta^f l_i$ is constant for map $F = \Phi$ or $F = \Gamma = (\Phi, \mathbf{u}^2)$. Then it is easy to see

$$\begin{aligned}&(I-\pi)\Delta^f v_F\\ &= 2(I-\pi)(\Sigma_{j_1<j_2}\nabla l_1 \wedge ... \wedge \nabla_{e_s}\nabla l_{j_1} \wedge ... \wedge \nabla_{e_s}\nabla l_{j_2} \wedge ...\nabla l_m)\\ &+ tr(\widehat{Ric^f}),\end{aligned} \tag{5.22}$$

where $tr(\widehat{Ric^f})$ is the trace over the space spanning by ∇l_i.

Lemma 5.8. *There exists a subset $E_{\epsilon,l} \subseteq B^{n-k}(1) \times [0,1]$, which depends only on ϵ, l and satisfies*

$$vol(E_{\epsilon,l}) \geq (1-\Psi)vol(B^{n-k}(1)\times[0,1]), \tag{5.23}$$

such that for any $(z,u)\in E_{\epsilon,l}$ *it holds*

$$\frac{1}{V_{\Phi,u}(z)}\int_{\Phi^{-1}(z)\cap U_u}|\Pi_{\Phi_z^{-1}}|^2<\Psi, \tag{5.24}$$

$$\frac{1}{V_\Gamma(z,u)}\int_{\Gamma^{-1}(z,u)}|\Pi_{\Phi_z^{-1}}|^2<\Psi, \tag{5.25}$$

$$\frac{1}{V_\Gamma(z,u)}\int_{\Gamma^{-1}(z,u)}|\Pi_{\Gamma^{-1}(z,u)}-u^{-1}g_{\Gamma^{-1}(z,u)}\otimes\nabla u|^2<\Psi. \tag{5.26}$$

Proof. Let ϕ be a cut-off function with support in $B_p(3)$ as constructed in Lemma 1.4. Note that v_Φ is almost 1 almost everywhere in U_u by the Hessian estimates in (5.10). Then by (5.22), we have

$$\begin{aligned}&\int_{U_u}(|v|^2+\eta)^{-\frac12}|\pi(\nabla v)|^2e^{-f}d\mathrm{v}\\&\le\int_{B_p(3)}|v_\Phi||(I-\pi)\Delta^f v_\Phi|e^{-f}d\mathrm{v}+\int_{B_p(3)}\phi\Delta^f((|v_\Phi|^2+\eta)^{\frac12}-1)e^{-f}d\mathrm{v}\\&<\Psi+\int_{B_p(3)}|\Delta^f\phi||(|v_\Phi|^2+\eta)^{\frac12}-1|e^{-f}d\mathrm{v}.\end{aligned}$$

By (5.21), it follows

$$\begin{aligned}&\int_{U_u}|v_\Phi||\Pi_{\Phi^{-1}(z)}|^2e^{-f}d\mathrm{v}\\&\le\lim_{\eta\to0}\int_{U_u}(|v|^2+\eta)^{-\frac12}|\pi(\nabla v)|^2e^{-f}d\mathrm{v}<\Psi.\end{aligned} \tag{5.27}$$

On the other hand, by the coarea formula, we have

$$\int_{B^{n-k}(1)}\int_{\Phi^{-1}(z)\cap U_u}|\Pi_{\Phi^{-1}(z)}|^2e^{-f}d\mathrm{v}=\int_{U_u}|v_\Phi||\Pi_{\Phi^{-1}(z)}|^2e^{-f}d\mathrm{v}.$$

Thus (5.24) follows from (5.27) immediately. Again by the coarea formula we get (5.25) from (5.24). (5.26) can be also obtained by using the same argument above to the map Γ (cf. Theorem 3.7 in [CCT]).

□

Completion of Proof of Proposition 5.5. We will finish the proof of Proposition 5.5 by applying the Gauss-Bonnet formula to an appropriate level set of Γ. In case $k=2$, by Lemma 5.7, we see that there exists (z,u) (u is close to 1) such that

$$|2\pi t-\frac{1}{u}V_\Gamma(z,u)|<\Psi,$$

where t is the radius of X. Note that X is a circle here. On the other hand, applying the Guass-Bonnet formulam to $\Phi^{-1}(z)\cap U_u$, we have

$$\int_{\Gamma^{-1}(z,u)} H + \int_{\Phi^{-1}(z)\cap U_u} K = 2\pi\chi(\Phi^{-1}(z)\cap U_u),$$

where K and H are Gauss curvature and mean curvature of $\Phi^{-1}(z)\cap U_u$ and $\Gamma^{-1}(z,u)$, respectively. By (5.24) and (5.7) together with the Gauss-Coddazzi equation, we see that

$$|\int_{\Phi^{-1}(z)\cap U_u} K| < \Psi.$$

Also we get from (5.26),

$$|\int_{\Gamma^{-1}(z,u)} H - \frac{1}{u}V_\Gamma(z,u)| < \Psi.$$

Thus t is close to $\chi(\Phi^{-1}(z)\cap U_u)$ which is an integer. The non-collapsing condition implies that $\chi(\Phi^{-1}(z)\cap U_u)$ is not zero. So $t > 1-\Psi$. As a consequence, the volume of ball $B(1)\subset\mathbb{R}^{n-1}\times C(X)$ is close to one of a unit flat ball. Hence by Remark 5.2, we see that $\mathrm{vol}(B_p(1))$ is close to $\mathrm{vol}(B_0(1))$. Therefore, we prove that $B_p(1)$ is close to $B_0(1)$ by Corollary 4.8.

In case $k=3$, we see that there exists (z,u) in Lemma 5.7 such that

$$|V_\Gamma(z,u) - \mathrm{vol}(X)| < \Psi, \tag{5.28}$$

as u is close to 1. On the other hand, by the Gauss-Bonnet formula, we have

$$\int_{\Gamma^{-1}(z,u)} K = 2\pi\chi(\Gamma^{-1}(z,u)). \tag{5.29}$$

Since by (5.25) and (5.7) together with the Gauss-Coddazzi equation,

$$\int_{\Gamma^{-1}(z,u)} |R_{\Phi^{-1}(z)}| < \Psi,$$

where $R_{\Phi^{-1}(z)}$ is the curvature tensor of the submanifold $\Phi^{-1}(z)$, (5.26) implies that

$$|\int_{\Gamma^{-1}(z,u)} K - V_\Gamma(z,u)| < \Psi.$$

By (5.29), it follows

$$V_\Gamma(z,u) > 4\pi - \Psi,$$

since the Euler number is even. Thus by (5.28), we get

$$\mathrm{vol}(B_p(1)) > \mathrm{vol}(B(1)) - \Psi > \mathrm{vol}(B_0(1)) - \Psi.$$

As a consequence, the volume $B_p(1)$ is close to one of $B_0(1)$. Therefore, we also prove that $B_p(1)$ is close to $B_0(1)$ by Corollary 4.8.

□

Proof of Theorem5.4. Let ϵ be a small number and $\delta = \delta(\epsilon)$ the constant determined in Proposition 5.5. For any $y \in Y$, we chose a sequence $p_j \in M_i$ which approaches to y. Define a subset in $B_{p_j}(2) \subset M_i$ for $\theta \leq \tau$ by

$$Q_i(\theta) = \{q \in B_{p_j}(2) | \frac{1}{\mathrm{vol}(B_q(s))} \int_{B_q(s)} |Rm(g_i)| \geq \delta s^{-2}, \ \exists \ s \leq \theta\}. \tag{5.30}$$

Denote the limit of $Q_i(\theta)$ in Y by $Q(\theta)$. We claim

Claim 5.9.

$$B_y(1) \subseteq \mathcal{R}_{3\epsilon} \cup Q(\theta) \cup \mathcal{S}_{n-4}. \tag{5.31}$$

Suppose that the claim is not true. Then there exist a point $z \bar{\in} \mathcal{R}_{3\epsilon} \cup Q(\theta) \cup \mathcal{S}_{n-4}$ and a tangent cone T_zY which is $\mathbb{R}^{n-k} \times C(X)$ for $k = 2$ or 3 and $d_{GH}(B_{z_\infty}(1), B_0(1)) > 3\epsilon$, where $z_\infty \cong z$. Thus there is a sequence r_i approaching 0 such that $(Y, \frac{d}{r_i}; z) \to T_zY$ in Gromov-Hausdorff topology. Hence for large enough i, we have

$$d_{GH}(B_z(r_i), B_0(r_i)) \geq 3\epsilon r_i,$$

$$d_{GH}(B_z(lr_i), B_{(0,x)}(lr_i)) \leq \frac{1}{2} r_i \eta,$$

where $\eta = \eta(\epsilon) << 1$ and $l = l(\epsilon) >> 1$ are both determined in Proposition 5.5. For fixed i in the above inequalities, we take j large enough and choose a point $p_j \in M_j$ such that

$$d_{GH}(B_{p_j}(r_i), B_0(r_i)) \geq 2\epsilon r_i \tag{5.32}$$

and

$$d_{GH}(B_{p_j}(lr_i), B_{(0,x)}(lr_i)) \leq r_i \eta. \tag{5.33}$$

Applying Proposition 5.5 to the manifold M_j with rescaling metric $\frac{g_i}{r_i}$ together with the condition (5.32), we get

$$d_{GH}(B_{p_j}(r_i), B_0(r_i)) \leq \epsilon r_i. \tag{5.34}$$

But this is impossible by (5.32). The claim is proved.

Let $Q = \bigcap_\theta Q(\theta)$. Then $B_y(1) \setminus \mathcal{R}_{2\epsilon} \subseteq Q \cup \mathcal{S}_{n-4}$. Now we only need to estimate $\mathcal{H}^{n-2p}(Q)$. Given j, by Vitali covering lemma, there is a collection of disjoint balls $B_{q_j}(s_j)$ with

$$\frac{1}{vol(B_{q_j}(s_j))} \int_{B_{q_j}(s_j)} |Rm(g_j)| \geq \delta s_j^{-2}$$

such that $\bigcup B_{q_j}(5s_j) \supseteq Q_i(\theta)$. Using the Hölder inequality, we see

$$\Sigma s_j^{n-2p} \leq \frac{c(\Lambda, v, A)}{\delta^p}. \tag{5.35}$$

It follows

$$\mathcal{H}_\theta^{n-2p}(Q) \leq \frac{c(\Lambda, v, A)}{\delta^p}.$$

Let θ approach 0, we get $\mathcal{H}_{n-2p}(Q) < \infty$. Since dim $\mathcal{S}_k \leq k$, we finish the proof of Theorem 5.4. □

6. Structure of singular set II: Case of Kähler metrics

In this section, we study the limit space for a sequence of Kähler metrics arising from solutions of certain complex Monge-Ampère equations for the existence problem of Kähler-Ricci soliton on a Fano manifold via the continuity method [TZ1], [TZ2]. We assume that (M, g) is a compact Kähler manifold with positive first Chern class $c_1(M) > 0$ (namely, M is Fano), where Kähler form ω_g of g is in $2\pi c_1(M)$. Then there exists a Ricci potential h of the metric g such that

$$\mathrm{Ric}(g) - \omega_g = \sqrt{-1}\partial\overline{\partial}h, \ \int_M e^h \omega_g^n = \int_M \omega^n = V.$$

In [TZ1], Tian and Zhu considered a family of complex Monge-Ampère equations for Kähler potentials ϕ on M,

$$\det(g_{i\bar{j}} + \phi_{i\bar{j}}) = \det(g_{i\bar{j}})e^{h-\theta_X - X(\phi) - t\phi}, \tag{6.1}$$

where $t \in [0, 1]$ is a parameter and θ_X is a real valued potential of a reductive holomorphic vector field on M which is defined

$$\bar{\partial}\theta_X = i_X\omega_g, \ \int_M e^{\theta_X}\omega_g^n = V,$$

according to the choice of g with K_X-invariant. The equations (6.1) are equal to

$$\mathrm{Ric}(\omega_\phi) - L_X\omega_\phi = t\omega_\phi + (1-t)\omega_g. \tag{6.2}$$

Thus ω_ϕ defines a Kähler-Ricci soliton if ϕ is a solution of (6.1) at $t = 1$. It was proved the set I of t for which (6.1) is solvable is open [TZ1]. In the other words, there exists $T \leq 1$ such that $I = [0, T)$. (6.2) implies

$$\mathrm{Ric}(\omega_\phi) + \sqrt{-1}\partial\overline{\partial}(-\theta_X(\phi)) \geq t\omega_\phi, \tag{6.3}$$

where $\theta_X(\phi) = \theta_X + X(\phi)$ is a potential of X associated to ω_ϕ, which is uniformly bounded [Zh].

Lemma 6.1. $|\overline{\partial}(\theta_X + X(\phi))| = |X|_{\omega_\phi}$ *and* $\Delta_{\overline{\partial}}(\theta_X(\phi))$ *are both uniformly bound by* $C(M, \omega, X)$*, where* $\Delta_{\overline{\partial}} = \frac{1}{2}\Delta$ *is a* $\overline{\partial}$*-Lapalace operator associated to* ω_ϕ*.*

Proof. We will use the maximum principle to prove the lemma. First we recall that $\theta_X(\phi)$ satisfies an identity [Fu],

$$\Delta_{\overline{\partial}}[\theta_X(\phi)] + \theta_X(\phi) + X(h) = 0,$$

where h is a Ricci potential of Kähler form ω_ϕ at t. Note that

$$h = \theta_X(\phi) + (t-1)\phi$$

by (6.2). Thus $\theta_X(\phi)$ satisfies

$$\Delta_{\overline{\partial}}[\theta_X(\phi)] + |\overline{\partial}\theta_X(\phi)|^2 + \theta_X(\phi) = (1-t)X(\phi). \tag{6.4}$$

By the Bochner formula, one sees

$$\begin{aligned}
&\Delta_{\overline{\partial}}(|\overline{\partial}\theta_X(\phi)|^2)\\
&= |\nabla\overline{\nabla}\theta_X(\phi)|^2 + 2\mathrm{re}(\langle\overline{\partial}\theta_X(\phi), \overline{\partial}\Delta_{\overline{\partial}}\theta_X(\phi)\rangle) + \mathrm{Ric}(\overline{\partial}\theta_X(\phi), \overline{\partial}\theta_X(\phi))
\end{aligned}$$

It follows

$$\begin{aligned}
&(\Delta_{\overline{\partial}} + X)(|\overline{\partial}\theta_X(\phi)|^2)\\
&= |\nabla\overline{\nabla}\theta_X(\phi)|^2 + 2\mathrm{re}(\langle\overline{\partial}\theta_X(\phi), \overline{\partial}(\Delta_{\overline{\partial}}\theta_X(\phi) + |\overline{\partial}\theta_X(\phi)|^2)\rangle)\\
&+ (\mathrm{Ric} - \nabla\overline{\nabla}\theta_X(\phi))(\overline{\partial}\theta_X(\phi), \overline{\partial}\theta_X(\phi)).
\end{aligned}$$

Thus by (6.4), we get

$$(\Delta_{\overline{\partial}} + X)(|\overline{\partial}\theta_X(\phi)|^2) = |\nabla\overline{\nabla}\theta_X(\phi)|^2 - t|\overline{\partial}\theta_X(\phi)|^2 - (1-t)|X|_g^2. \tag{6.5}$$

Note that

$$|\nabla\overline{\nabla}\theta_X(\phi)|^2 \geq \frac{(\Delta_{\overline{\partial}}\theta_X(\phi))^2}{n} \geq \frac{(|\overline{\partial}\theta_X(\phi)|^2 - C_1)^2}{n},$$

where $C_1 = \max_M\{|\theta_X(\phi) - (1-t)X(\phi)|\}$. Apply the maximum principle to $|\overline{\partial}\theta_X(\phi)|^2$ in (6.5), we derive at a maximal point of $|\overline{\partial}\theta_X(\phi)|^2$,

$$0 \geq \frac{1}{n}(|\overline{\partial}\theta_X(\phi)|^2 - C_1)^2 - t|\overline{\partial}\theta_X(\phi)|^2 - C_2. \tag{6.6}$$

Therefore, the gradient estimate of $\theta_X(\phi)$ follows from the above inequality immediately. By (6.4), we also get the $\overline{\partial}$-Lapalace estimate of $\theta_X(\phi)$. □

By Lemma 6.1 and Theorem 5.1, we prove

Theorem 6.2. *For any sequence of Kähler metrics g_{t_i} associated to solutions ϕ_{t_i} of equations (6.1) at $t = t_i \in I$, there exists a subsequence which converge to a limit metric space Y in Gromov-Hausdorff topology. Moreover, $\mathcal{S}(Y) = \mathcal{S}_{2n-2}$. In particular, the complex codimension of singularities of Y is at least 1.*

Proof. We suffice to

$$\mathrm{vol}_{g_t}(B_p(1)) \geq v > 0. \ \forall\ p \in M. \tag{6.7}$$

But this is just a consequence of application of the Volume Comparison Theorem 1.2 since the diameter of g_t is uniformly bounded by a result of Mabuchi [Ma]. □

In a special case $t_i \to 1$ when $I = [0, 1)$ in Theorem 6.2, we can strengthen Theorem 6.2 as follows.

Theorem 6.3. *Let g_{t_i} be a sequence of Kähler metrics in Theorem 6.2 with $t_i \to 1$. Then $\mathcal{S}(Y) = \mathcal{S}_{2n-4}$. In particular, the complex codimension of singularities of Y is at least 2.*

$I = [0, 1)$ can be guaranteed when the modified Mabuchi K-energy is bounded from below and X is an extremal holomorphic vector field which determined by the modified Futaki-invariant [TZ2]. This can be proved following an argument by Futaki for the study of almost Kähler-Einstein metric under an assumption that the Mabuchi K-energy is bounded from below on a Fano manifold [Fu]. Thus as a corollary of Theorem 6.3, we have

Corollary 6.4. *Suppose that the modified K-energy is bounded from below on a Fano manifold. There exists a subsequence of weak almost Kähler-Ricci solitons on M which converge to a limit metric space Y in Gromov-Hausdorff topology. Moreover, the complex codimension of singularities of Y is at least 2.*

It may be useful to introduce a more general sequence of Kähler metrics than one in Theorem 6.3 inspired by a recent work of Wang and Tian [WT].

Definition 6.5. *We call a sequence of Kähler metrics (M_i, J_i, g_i) weak almost Kähler-Ricci solitons if there are uniform constants Λ and A such that*

$$i)\ Ric(g_i) + \nabla\overline{\nabla} f_i \geq -\Lambda^2 g_i,\ \nabla\nabla f_i = 0;$$
$$ii)\ \|\overline{\partial} f_i\|_{g_i} \leq A;$$
$$iii)\ lim_i \|Ric(g_i) - g_i + \nabla\overline{\nabla} f_i\|_{L^1(g_i)} \to 0.$$

Here f_i are some smooth functions and they define reductive holomorphic vector fields on Fano manifolds (M_i, J_i).

Lemma 6.6. *Let $\{g_{t_i}\}$ be a sequence of Kähler metrics in Theorem 6.2 with $t_i \to 1$. Then $\{g_{t_i}\}$ is a sequence of weak almost Kähler-Ricci solitons on M.*

Proof. By Lemma 6.1, it suffice to check the condition iii) in Definition 6.5. In fact, we have

$$\begin{aligned}
&\int_M |\mathrm{Ric}(\omega_\phi) - \sqrt{-1}\partial\bar\partial\theta_X(\phi) - \omega_\phi| \\
&\le \int_M |\mathrm{Ric}(\omega_\phi) - \sqrt{-1}\partial\bar\partial\theta_X(\phi) - t\omega_\phi| + n(1-t)\mathrm{vol}(M) \\
&= \int_M (\mathrm{Ric}(\omega_\phi) - \sqrt{-1}\partial\bar\partial\theta_X(\phi) - t\omega_\phi) \wedge \frac{\omega_\phi^{n-1}}{(n-1)!} + n(1-t)\mathrm{vol}(M) \\
&= 2n(1-t)\mathrm{Vol}(M) \to 0.
\end{aligned}$$

□

We now begin to prove the following main result in this section.

Theorem 6.7. *Let (M_i, g_i) be a sequence of weak almost Kähler-Ricci solitons. Suppose that there exists a point p_i at each M_i such that*

$$vol_{M_i}(B_{p_i}(1)) \ge v > 0. \tag{6.8}$$

Then there exists a subsequence of $(M_i, g_i; p_i)$ which converge to a limit metric space Y_∞ in pointed Gromov-Hausdorff topology. Moreover $\mathcal{S}(Y_\infty) = \mathcal{S}_{2n-4}$. In particular, the complex codimension of singularities of Y is at least 2.

We need a lemma of ϵ-regularity for the tangent cone to prove Theorem 6.7.

Lemma 6.8. *For any $\mu_0, \epsilon > 0$, there exist small numbers $\delta = \delta(v, \epsilon, n)$, $\eta = \eta(v, \epsilon, n)$, $\tau = \tau(v, \epsilon, n)$ and a big number $l = l(v, \epsilon, n)$ such that if a Kähler manifold (M^n, g) satisfies*

$$\begin{aligned}
&i)\ Ric_M^f(g) > -(n-1)\tau^2 g, \nabla\nabla f = 0, \\
&ii)\ vol_g(B_p(1)) \ge \mu_0, \\
&iii)\ |\nabla f| < \tau, \\
&iv)\ \frac{1}{vol(B_p(2))} \int_{B_p(2)} |Ric(g) + \nabla\overline{\nabla} f| dV_g < \delta, \\
&v)\ d_{GH}(B_p(l), B_{(0,x)}(l)) < \eta,
\end{aligned}$$

where $B_{(0,x)}(l)$ is a l-radius ball in cone $\mathbb{R}^{2n-2} \times C(X)$ centered at the vertex $(0, x)$ for some metric space X, then

$$d_{GH}(B_p(1), B(1)) < \epsilon. \tag{6.9}$$

Proof. The proof of Lemma 6.8 is a modification to one of Proposition 5.5. Note that X is a circle of radius t in present case. It suffices to show that t is close to 2π by Lemma 4.8. Let $\Phi=(h_1,...,h_{2n-2})$ and $\Gamma=(\Phi,\mathbf{u})$ be two maps constructed in Proposition 5.5. By Proposition 8.4 in Appendix 2, we may also assume

$$\int_{B_p(3)}|\nabla h_{n-1+i}-\mathbf{J}\nabla h_i|^2<\Psi(\tau,\epsilon,\frac{1}{l};v). \tag{6.10}$$

We shall compute the differential characteristic $\widehat{c_{1,\nabla}}$ of tangent bundle $(TM,\,\nabla)$ restricted on $\Gamma^{-1}(z,u)=\Phi^{-1}(z)\cap U_u$ with fixed z (cf. [C3]), where ∇ is the Levi-Civita connection on TM and (z,u) is a regular point of Γ such that both Lemma 5.7 and Lemma 5.8 hold. It is easy to see that by the coarea formula and the condition iv), the set

$$\begin{aligned} D=\{z|\ &\Phi^{-1}(z)\cap U_u \text{ is a regular surface in } M \text{ and}\\ &\int_{\Phi^{-1}(z)\cap U_u}|\mathrm{Ric}(g)+\nabla\overline{\nabla}f|<c\delta\}\end{aligned} \tag{6.11}$$

has a positive volume in $\mathbb{R}^{2n-2}$ for some constant c which depends only on n.

For each $z\in D$, we have the estimate

$$\begin{aligned} &|\int_{\Phi^{-1}(z)\cap U_u}\mathrm{Ric}\ (\omega_g)|\\ &\le\int_{\Phi^{-1}(z)\cap U_u}|\mathrm{Ric}(g)+\nabla\overline{\nabla}f|+|\int_{\Phi^{-1}(z)\cap U_u}\sqrt{-1}\partial\overline{\partial}f|\\ &\le c\delta+\int_{\Gamma^{-1}(z,u)}|\nabla f|\le c\delta+\mathrm{vol}(\Gamma^{-1}(z,u))\tau.\end{aligned} \tag{6.12}$$

Since

$$\int_{\Gamma^{-1}(z,u)}\widehat{c_{1,\nabla}}=\int_{\Phi^{-1}(z)\cap U_u}\mathrm{Ric}\ (\omega_g),\ \mathrm{mod}\ \mathbb{Z},$$

we get

$$\int_{\Phi^{-1}(z)\cap U_u}\widehat{c_{1,\nabla}}=\Psi,\ \mathrm{mod}\ \mathbb{Z}. \tag{6.13}$$

To compute the left term of (6.13), we will decompose the tangent bundle $(TM,\,\nabla)$ over $\Gamma^{-1}(z,u)$ as follows.

By our construction of the map Γ, using the coarea formula, we may assume that

$$i)\ \int_{\Gamma^{-1}(z,u)} |\langle \nabla h_i, \nabla h_j\rangle - \delta_{ij}| < \Psi,$$
$$ii)\ \int_{\Gamma^{-1}(z,u)} |\text{hess } h_i| < \Psi,$$
$$iii)\ \int_{\Gamma^{-1}(z,u)} |\langle \nabla \mathbf{u}^2, \nabla h_j\rangle| < \Psi,$$
$$iv)\ \int_{\Gamma^{-1}(z,u)} |\nabla\langle \nabla \mathbf{u}^2, \nabla h_j\rangle| < \Psi.$$

Since $\Gamma^{-1}(z,u)$ is one dimensional manifold with bounded length, the conditions i- ii) and iii-iv) imply

$$|\langle \nabla h_i, \nabla h_j\rangle - \delta_{ij}| \text{ and } |\langle \nabla \mathbf{u}^2, \nabla h_j\rangle|$$

are both small on $\Gamma^{-1}(z,u)$, respectively. Moreover, applying the coarea formula to (6.10) together with the above condition ii), we also get

$$|\nabla h_{n-1+i} - \mathbf{J}\nabla h_i| < \Psi.$$

Hence by using the Gram-Schmidt process, we obtain $(2n-1)$ orthogonal sections of TM over $\Gamma^{-1}(z,u)$,

$$e_i, \mathbf{J}(e_i)\ (1 \le i \le n-1), \mathbf{N}$$

from sections ∇h_i $(1 \le i \le n-1)$, $\nabla \mathbf{u}$. Denote $\mathbb{E}$ to be the sub-bundle spanning by $e_i, \mathbf{J}(e_i)$ and decompose TM into

$$TM = \mathbb{E} \oplus \mathbb{E}^\perp \tag{6.14}$$

where $\mathbb{E}^\perp$ is the orthogonal complement of $\mathbb{E}$. We introduce a Whitney sum connection ∇' on TM over $\Gamma^{-1}(z,u)$ by combining two projection connections on $\mathbb{E}$ and $\mathbb{E}^\perp$, which are both induced by ∇. Then by the condition ii), it is easy to show

$$\int_{\Gamma^{-1}(z,u)} |\nabla - \nabla'| < \Psi, \tag{6.15}$$

where $\nabla - \nabla'$ is regarded as a 1-form on $\mathrm{End}(TM)$. Also we can introduce another connection ∇'' which is flat on $\mathbb{E}$. Namely, ∇'' satisfies

$$\nabla''(e_i) = \nabla''(\mathbf{J}(e_i)) = 0.$$

Similar to (6.15), we have

$$\int_{\Gamma^{-1}(z,u)} |\nabla'' - \nabla'| < \Psi. \tag{6.16}$$

Therefore, combining (6.15) and (6.16), we derive

$$|(\widehat{c_{1,\nabla''}} - \widehat{c_{1,\nabla}})(\Gamma^{-1}(z,u))| << 1.$$

On the other hand, by the flatness of ∇'' on $\mathbb{E}$ over $\Gamma^{-1}(z,u)$, the quantity $2\pi\widehat{c_{1,\nabla''}}(\Gamma^{-1}(z,u))$ is just equal to the holonomy of the connection around $\Gamma^{-1}(z,u)$ (measured by angle),

$$2\pi\widehat{c_{1,\nabla''}}(\Gamma^{-1}(z,u)) = \int_{\Gamma^{-1}(z,u)} \langle \nabla''_X \mathbf{N}, \mathbf{JN}\rangle, \tag{6.17}$$

where X is the unit tangent vector of $\Gamma^{-1}(z,u)$. Thus by the choice of $\mathbf{N}$ together with (6.15), (6.16) and (5.26), we see that the angle is close to the length of $\Gamma^{-1}(z,u)$. By (6.13), it follows that $\frac{\mathrm{vol}(\Gamma^{-1}(z,u))}{2\pi}$ is close to zero modulo integers. Hence, the non-collapsing of $B_{(0,x)}(1)$ implies that $\mathrm{vol}(\Gamma^{-1}(z,u))$ is close to 2π. Consequently, we prove that t is close to 2π by (5.19) in Lemma 5.7.

□

Proof of Theorem 6.7. By the Volume Comparison Theorem 1.2, for any $r \leq 1$, we have

$$\mathrm{vol}_{g_i}(\mathrm{vol}(B_p(r)) \geq \lambda_0 r^n, \ \forall\, p \in \ M_i,$$

where λ_0 depends only on the constants Λ, A, v in Definition 6.5. Thus by Gromov's compactness theorem [Gr], there exists a subsequence of $(M_i, g_i; p_i)$ which converge to a metric space Y_∞ in pointed Gromov-Hausdorff topology. In the remaining, we show that $\mathcal{S}(Y_\infty) = \mathcal{S}_{2n-4}$. We will use an argument by the contradiction. On the contrary, for a ball $B_y(1) \subset Y$, by Proposition 8.5 in Appendix 2, there exists a point $z \in S \cap B_y(1) \subsetneqq S_{2n-4}$ and there exists a sequence $\{r_i\}$ $(r_i \to 0)$ such that $(Y, \frac{d}{r_i^2}; z)$ converge a tangent cone $T_zY = \mathbb{R}^{2n-2} \times C(X)$. This implies that exists an $\epsilon > 0$ such that the unit metric ball $B_{z_\infty}(1) \subset T_zY$ centered at $z_\infty \cong z$ satisfies

$$d_{GH}(B_{z_\infty}(1), B(1)) > 2\epsilon, \tag{6.18}$$

and for any $l >> 1$ and $\epsilon << 1$ one can choose sufficiently large numbers i and k such that

$$\begin{aligned} &d_{GH}(\hat{B}_{z_k}(1), B(1)) > \epsilon, \\ &d_{GH}(\hat{B}_{z_k}(l), B_{(0,x)}(l)) < \eta, \end{aligned} \tag{6.19}$$

where $z_k \in M_k \to z \in Y$ as $k \to \infty$, and $\hat{B}_{z_k}(1)$ and $\hat{B}_{z_k}(l)$ are two balls with radius 1 and l respectively in $(M_k, \frac{g_k}{r_i^2}) = (M_k, \hat{g}_k)$. On the other hand, by using the Volume Comparison Theorem 1.2, for fixed i, we can choose

large enough k such that

$$\frac{r_i^2}{\mathrm{vol}(B_{z_k}(2r_i))}\int_{B_{z_k}(2r_i)}|\mathrm{Ric}(g_k)-g_k+\nabla\overline{\nabla}f_k|dv_{g_k}<\frac{1}{2}\delta.$$

Since

$$\frac{r_i^2}{\mathrm{vol}(B_{z_k}(2r_i))}\int_{B_{z_k}(2r_i)}|g_k|dv_{g_k}\le c(n,C)r_i^2,$$

we have

$$\frac{1}{\mathrm{vol}(\hat{B}_{z_k}(2))}\int_{\hat{B}_{z_k}(2)}|\mathrm{Ric}(\hat{g}_k)+\nabla\overline{\nabla}f_k|dv_{\hat{g}_k}<\delta. \tag{6.20}$$

Hence, for large k, $(M_k,\hat{g}_k)$ satisfies the conditions i-v) in Lemma 6.8, and consequently, we get

$$d_{GH}(\hat{B}_{z_k}(1),B(1))<\epsilon,$$

which is a contradiction to (6.19). The theorem is proved. □

Theorem 6.3 follows from Theorem 6.7 with the help of Lemma 6.6 and the relation (6.7).

7. Appendix 1

This appendix is a discussion about how to use the technique of conformal transformation as in [TZh] to prove Theorem 6.2 and Theorem 6.3 in Section 6. We would like to emphasis on the different situation after the change of Ricci curvature by conformal transformation.

First, Theorem 6.2 can be proved by using the conformal technique. In fact, by the formula of Ricci curvature for conformal metric $e^{2u}g$,

$$\begin{aligned}&\mathrm{Ric}\ (e^{2u}g)\\ &=\mathrm{Ric}\ (g)-(n-2)(\mathrm{hess}\ u-du\otimes du)+(\Delta u+(n-2)|\nabla u|^2)g,\end{aligned}\tag{7.1}$$

the condition $\mathrm{Ric}\ _M^f(g)\ge -C$ implies that Ricci curvature $\mathrm{Ric}\ (e^{-\frac{2f}{n-2}}g)$ of conformal metric $e^{-\frac{2f}{n-2}}g$ is bounded from below if both ∇f and Δf are bounded. Thus by Lemma 6.1, we see that

$$\mathrm{Ric}\ (e^{\frac{2\theta_X(\phi_t)}{n-2}}g_t)$$

is uniformly bounded from below. Hence, Theorem 6.2 follows from Theorem 6.2 in [CC2] immediately.

Secondly, following the proof of Theorem 5.4 in [C3], Lemma 6.8 with an additional condition $v)$ $|\Delta f|<\tau$ can be proved by using the conformal change of the bundle metric. We note that the condition $v)$ can be guaranteed for the Kähler manifolds (M,g_t) in Theorem 6.3 with blowing-up

metrics. Thus by (7.1), Ricci curvature of blowing-up metric of $e^{\frac{2\theta_X(\phi_t)}{n-2}} g_t$ is almost positive.

For a Kähler manifold $(M, g, \mathbf{J})$, the $(1,0)$-type Hermitian connection ∇ on the holomorphic bundle (TM, h) is same as the Levi-Civita connection, where h is the Hermitian metric corresponding to g. Then $c_{1,\nabla}$ of (TM, h) is the same as the Ricci form of g. If we choose a Hermitian metric $e^{\psi} g$ for a smooth function ψ, then

$$\tilde{\nabla} = \nabla + \partial\psi$$

is the corresponding $(1,0)$-type Hermitian connection. It follows

$$F^{\tilde{\nabla}} = F^{\nabla} + d\partial\psi$$

and

$$\sqrt{-1}tr(F^{\tilde{\nabla}}) = \sqrt{-1}tr(F^{\nabla}) - n\sqrt{-1}\partial\bar{\partial}\psi, \tag{7.2}$$

where F^{∇} $(F^{\tilde{\nabla}})$ denotes the curvature of the connection ∇ $(\tilde{\nabla})$ on TM. Thus by putting $\psi = -\frac{2\pi}{n} f$ and using (7.2), we have

$$\widehat{c_{1,\tilde{\nabla}}}(\Gamma^{-1}(z,u)) = \int_{\Gamma^{-1}(z,u)} |\text{Ric }(\omega_g) + \sqrt{-1}\partial\bar{\partial} f|, \ \text{mod } \mathbb{Z}, \tag{7.3}$$

where the map Γ is defined as in Section 5 and Section 6 for the conformal metric $\tilde{g} = e^{-\frac{2f}{n-2}} g$. Thus $\widehat{c_{1,\tilde{\nabla}}}(\Gamma^{-1}(z,u))$ is small modulo integers. Moreover, by Theorem 3.7 in [CCT] (compared to Lemma 5.8 in Section 5) , it holds

$$\frac{1}{V_\Gamma(z,u)} \int_{\Gamma^{-1}(z,u)} |\Pi_{\Gamma^{-1}(z,u)} - u^{-1}\tilde{g}_{\Gamma^{-1}(z,u)} \otimes \nabla u|^2 < \Psi. \tag{7.4}$$

On the other hand, since the Ricci curvature of $\tilde{g}$ is almost positive, for the connection $\tilde{\nabla}$, we can follow the argument in proof of Theorem 5.4 [C3] to show that the quantity $2\pi\widehat{c_{1,\tilde{\nabla}}}(\Gamma^{-1}(z,u))$ is close to a holonomy of another perturbation connection $\tilde{\nabla}''$ of $\tilde{\nabla}$ around $\Gamma^{-1}(z,u)$ (also see the argument in proof of Lemma 6.8). The late is close to

$$\int_{\Gamma^{-1}(z,u)} \Pi_{\Gamma^{-1}(z,u)}.$$

Thus combining (7.3) and (7.4), we get

$$|\widehat{c_{1,\tilde{\nabla}}}(\Gamma^{-1}(z,u)) - \frac{\text{vol}(\Gamma^{-1}(z,u))}{2\pi}| < \Psi.$$

It follows that the diameter of section X in two dimensional cone $C(X)$ with rescaled cone metric is close to 2π. Thus the Gromov-Hausdorff distance between $B_p(1)$ and $B_{(0,x)}(1)$ both with rescaled metrics is close to zero. By Theorem 9.69 in [C1], we prove Lemma 6.8 with the additional condition v). Theorem 6.3 follows from applying Lemma 6.8 to the sequence $\{(M, g_t)\}$

($t \to 1$) with blowing-up metrics, for details to see the proof of Theorem 6.7 in the end of Section 6.

8. Appendix 2

In this appendix, we prove (6.10) in Section 6. We need several lemmas. First, as an application of Lemma 2.5, we have

Lemma 8.1. *Under the conditions of Lemma 2.4, for a vector field X on $A_p(a,b)$ which satisfies*

$$|X|_{C^0(A_p(a,b))} \le D, \frac{1}{vol^{\,f}(A_p(a,b))}\int_{A_p(a,b)} |\nabla X|^2 dv^f < \delta, \tag{8.1}$$

there exists a f-harmonic function θ defined in $A_p(a_2,b_2)$ such that

(8.2)
$$\frac{1}{vol^{\,f}(A_p(a_2,b_2))}\int_{A_p(a_2,b_2)} |\nabla\theta - X|^2 dv^f < \Psi(\epsilon,\omega,\delta; A, a_1, b_1, a_2, a, b),$$

and

$$\begin{aligned} &\frac{1}{vol^{\,f} A_p(a_3,b_3)}\int_{A_p(a_3,b_3)} |hess\ \theta|^2 dv^f \\ &< \Psi(\epsilon,\omega,\delta; A, a_1, b_1, a_2, b_2, a_3, b_3, a, b), \end{aligned} \tag{8.3}$$

where $A_p(a_3,b_3)$ is an even smaller annulus in $A_p(a_2,b_2)$.

Proof. Let h be the f-harmonic function constructed in (2.12) in Section 2 and $\theta_1 = \langle X, \nabla h\rangle$. Then

$$\nabla\theta_1 = \langle\nabla X, \nabla h\rangle + \langle X, \text{hess } h\rangle,$$

It follows

$$\begin{aligned} &\int_{A_p(a_2,b_2)} |\nabla\theta_1 - X|^2 d\text{v}^f \\ &\le 2\int_{A_p(a_2,b_2)} (\langle\nabla X, \nabla h\rangle^2 d\text{v}^f + \langle X, \text{hess } h - g\rangle^2) d\text{v}^f. \end{aligned}$$

Thus by (8.1) and Lemma 2.5, we get

$$\frac{1}{\text{vol}^{\,f}(A_p(a_2,b_2))}\int_{A_p(a_2,b_2)} |\nabla\theta_1 - X|^2 d\text{v}^f < \Psi. \tag{8.4}$$

Let θ be a solution of equation,

$$\Delta^f\theta = 0, \quad \text{in } A_p(a_2,b_2), \tag{8.5}$$

with $\theta = \theta_1$ on $\partial A_p(a_2, b_2)$. Then

$$\begin{aligned}&\int_{A_p(a_2,b_2)} (\langle \nabla\theta - \nabla\theta_1, X\rangle + (\theta-\theta_1)\text{div } X) d\mathrm{v}^f \\ &= \int_{A_p(a_2,b_2)} \text{div } ((\theta-\theta_1)X) dv^f = \int_{A_p(a_2,b_2)} (\theta-\theta_1)\langle \nabla f, X\rangle d\mathrm{v}^f.\end{aligned}$$

It follows

$$\int_{A_p(a_2,b_2)} \langle \nabla\theta - \nabla\theta_1, X\rangle d\mathrm{v}^f < \Psi. \tag{8.6}$$

On the other hand, since

$$\int_{A_p(a_2,b_2)} \langle \nabla\theta_1 - \nabla\theta, \nabla\theta\rangle d\mathrm{v}^f = \int_{A_p(a_2,b_2)} (\theta-\theta_1)\Delta^f\theta d\mathrm{v}^f = 0,$$

we have

$$\int_{A_p(a_2,b_2)} |\nabla\theta|^2 d\mathrm{v}^f = \int_{A_p(a_2,b_2)} \langle \nabla\theta, \nabla\theta_1\rangle d\mathrm{v}^f.$$

By the Hölder inequality, we get

$$\int_{A_p(a_2,b_2)} |\nabla\theta|^2 d\mathrm{v}^f \le \int_{A_p(a_2,b_2)} |\nabla\theta_1|^2 d\mathrm{v}^f < C.$$

Hence,

$$\begin{aligned}&\int_{A_P(a_2,b_2)} \langle \nabla\theta - X\rangle^2 d\mathrm{v}^f \\ &= \int_{A_p(a_2,b_2)} (|\nabla\theta|^2 + |X|^2 - 2\langle\nabla\theta, X\rangle) dv^f \\ &= \int_{A_p(a_2,b_2)} (\langle\nabla\theta, \nabla\theta_1\rangle + |X|^2 - 2\langle\nabla\theta, X\rangle) d\mathrm{v}^f \\ &= \int_{A_p(a_2,b_2)} (\langle\nabla\theta_1 - X, \nabla\theta\rangle + \langle X, X - \nabla\theta_1\rangle + \langle X, \nabla\theta_1 - \nabla\theta\rangle) d\mathrm{v}^f.\end{aligned} \tag{8.7}$$

Therefore, combining (8.1) and (8.6), we derive (8.2) immediately.

To get (8.3), we choose a cut-off function which is ϕ supported in $A_p(a_2, b_2)$ with bounded gradient and f-Lapalace as in Lemma 1.4 in Section 1. Then by the Bochner identity, we have

$$\int_{A_p(a_2,b_2)} \frac{1}{2}\phi\Delta^f|\nabla\theta|^2 d\mathrm{v}^f = \int_{A_p(a_2,b_2)} \phi(|\text{hess } \theta|^2 + \text{Ric } (\nabla\theta, \nabla\theta)) d\mathrm{v}^f.$$

Since

$$\int_{A_p(a_2,b_2)} \frac{1}{2}\phi\Delta^f|X|^2 d\mathrm{v}^f = -\int_{A_p(a_2,b_2)} \langle\nabla\phi, \langle X, \nabla X\rangle\rangle d\mathrm{v}^f,$$

we obtain

$$\begin{aligned}\int_{A_p(a_2,b_2)} \phi(|\text{hess } \theta|^2 d\text{v}^f &< \int_{A_p(a_2,b_2)} \frac{1}{2}\phi\Delta^f(|\nabla\theta|^2 - |X|^2)d\text{v}^f \\ &+ \Psi(\epsilon,\omega,\delta;A,a_1,b_1,a_2,b_2,a_3,b_3,a,b).\end{aligned} \tag{8.8}$$

Therefore, using integration by parts, we derive (8.3) from (8.2). □

Next, we generalize Proposition 3.6 to the case without the assumption of the existence of an almost line.

Lemma 8.2. *Let (M,g) be a Riemannian manifold which satisfies (3.2). Let h^+ be a f-harmonic function which satisfies*

$$|\nabla h^+| \le c(n,\Lambda,A), \tag{8.9}$$

$$\frac{1}{vol^f(B_p(1))}|\int_{B_p(1)} |\nabla h^+|^2 - 1| < \delta, \tag{8.10}$$

$$\frac{1}{vol^f(B_p(1))}\int_{B_p(1)} |hess\ h^+|^2 < \delta. \tag{8.11}$$

Then there exists a $\Psi(\delta;A,\Lambda,n)$ Gromov-Hausdorff approximation from $B_p(\frac{1}{8})$ to $B_{(0\times x)}(\frac{1}{8}) \subset \mathbb{R}\times X$.

The proof of Lemma 8.2 depends on the following fundamental lemma which is in fact a consequence of Theorem 16.32 and Lemma 8.17 in [C1].

Lemma 8.3. *Under the condition (3.2), for a f-harmonic function h^+ which satisfies (8.9), (8.10) and (8.11) in $B_p(1)$, there exists a Lipschitz function ρ in $B_p(\frac{1}{4})$ such that $|h^+ - \rho| < \Psi$ and*

$$||\rho(z) - t| - d(z,\rho^{-1}(t))| < \Psi. \tag{8.12}$$

Proof. First, we notice that the following Poincaré inequality holds for any C^1-function h,

$$\begin{aligned}&\frac{1}{\text{vol}^f(B_p(\frac{1}{2}))}\int_{B_p(\frac{1}{2})} |h-a|^2 d\text{v}^f \\ &\le c(n,\Lambda,A)\frac{1}{\text{vol}^f(B_p(1))}\int_{B_p(1)} |\nabla h|^2 d\text{v}^f,\end{aligned} \tag{8.13}$$

where

$$a = \frac{1}{\text{vol}^f(B_p(\frac{1}{2}))}\int_{B_p(\frac{1}{2})} h d\text{v}^f.$$

This is in fact a consequence of Lemma 3.3 by applying the function e to $|\nabla h|^2$, because

$$\begin{aligned}
&\frac{1}{\mathrm{vol}^f(B_p(\frac{1}{2}))}\int_{B_p(\frac{1}{2})}|h(x)-a|^2 d\mathrm{v}^f\\
&=\frac{1}{\mathrm{vol}^f(B_p(\frac{1}{2}))}\int_{B_p(\frac{1}{2})}d\mathrm{v}_x^f[\frac{1}{\mathrm{vol}^f(B_p(\frac{1}{2}))}\int_{B_p(\frac{1}{2})}(h(x)-h(y))d\mathrm{v}_y^f]^2\\
&\le\frac{1}{\mathrm{vol}^f(B_p(\frac{1}{2}))}\int_{B_p(\frac{1}{2})}\frac{1}{\mathrm{vol}^f(B_p(\frac{1}{2}))}\int_{B_p(\frac{1}{2})}(h(x)-h(y))^2 d\mathrm{v}_x^f d\mathrm{v}_y^f\\
&\le\frac{1}{\mathrm{vol}^f(B_p(\frac{1}{2}))}\int_{B_p(\frac{1}{2})}\frac{1}{\mathrm{vol}^f(B_p(\frac{1}{2}))}\int_{B_p(\frac{1}{2})}\int_0^{d(x,y)}|\nabla h((\gamma(s))|^2\mathrm{v}_x^f d\mathrm{v}_y^f\\
&\le c(n,\Lambda,A)\frac{1}{\mathrm{vol}^f(B_p(1))}\int_{B_p(1)}|\nabla h|^2 d\mathrm{v}^f.
\end{aligned}$$

Thus by taking $h=|\nabla h^+|^2$, we get from (8.9)-(8.11),

$$\frac{1}{vol^f(B_p(\frac{1}{2}))}\int_{B_p(\frac{1}{2})}||\nabla h^+|^2-1|d\mathrm{v}^f<\Psi. \tag{8.14}$$

Next we apply Theorem 16.32 in [C1] to h^+ with the condition (8.9), (8.10) and (8.14). We suffice to check a doubling condition for the measure $d\mathrm{v}^f$ and an (ϵ,δ)-inequality. The (ϵ,δ)-inequality says, for any $\epsilon,\delta>0$ and two points $x,y\in M$ with $d(x,y)=r$, there exist $C_{\epsilon,\delta}$ and another two points x',y' with $d(x',x)\le\delta r$ and $d(y',y)\le\delta r$, respectively such that

$$F_{\phi,\epsilon}(z_1',z_2')\le\frac{C_{\epsilon,\delta}r}{\mathrm{vol}^f(B_{z_1}((1+\delta)(1+2\epsilon)r))}\int_{B_{z_1}((1+\delta)(1+2\epsilon)r)}\phi d\mathrm{v}^f, \tag{8.15}$$

where

$$F_{\phi,\epsilon}(x,y)=\inf\int_0^l\phi(c(s))ds,\ \forall\ \phi(\ge 0)\in\ C^0(M),$$

and the infimum takes among all curves from x to y with length $l\le(1+\epsilon)d(x,y)$. The doubling condition follows from Volume Comparison Theorem 1.2, and (ϵ,δ)-inequality follows from Volume Comparison Theorem 1.2 and the segment inequality in Lemma 3.3. Thus we can construct a Lipschitz function ρ from h^+ such that $|h^+-\rho|\le\Psi$. Moreover, by Lemma 8.17 in [C1], we get (8.12). □

Proof of Lemma 8.2. As in the proof of Proposition 3.6, we define $X=(h^+)^{-1}(0)$ and the map u by

$$u(q)=(h^+(q),x_q),$$

where x_q is the nearest point in X to q. To show that u is a Gromov-Hausdorff approximation, we shall use Lemma 3.2. In fact, by (8.12) in

Lemma 8.3, we see

$$||h^+(z) - t| - d(z, (h^+)^{-1}(t))| < \Psi. \tag{8.16}$$

Then instead of (3.1) by (8.16), Lemma 3.2 is still true since (8.11) holds [C2]. Hence the proof in Proposition 3.6 works for Lemma 8.2. □

Now we begin to prove (6.10) in Section 6. Let (M, g) be a Kähler manifold which satisfies (5.6). Let $B_p(l) \subset M$ and $B_{(0\times x)}(l) \subset \mathbb{R}^{2n-2} \times X$ be two l-radius distance balls as in Section 6. Then

Proposition 8.4. *Suppose that*

$$d_{GH}(B_p(l), B_{(0\times x)}(l)) < \eta. \tag{8.17}$$

Then either $B_p(\frac{1}{8})$ is close to an Euclidean ball in Gromov-Hausdorff topology or for a suitable choice of the orthogonal coordinates in $\mathbb{R}^{2n-2}$, the map Φ constructed in Section 5 satisfies

$$\frac{1}{vol^f B_p(1)} \int_{B_p(1)} |\nabla h_{n-1+i} - \mathbf{J}\nabla h_i|^2 dv^f < \Psi(\tau, \eta, \frac{1}{l}; v). \tag{8.18}$$

Proof. Roughly speaking, if the space spanned by ∇h_i is not almost $\mathbf{J}$ invariant, we can find a vector field nearly perpendicular to these ∇h_i, and it satisfies the condition (8.1) in Lemma 8.1. Then by Proposition 8.2, $B_p(1)$ will be almost split off along a new line. This implies that $B_p(\frac{1}{8})$ is close to an Euclidean ball.

Let V be a $(4n-4)$-dimensional line space spanned by $\nabla h_i, \mathbf{J}\nabla h_i$ with the L^2-inner product,

$$(b_i, b_j)_{L^2} = \int_{B_p(1)} \langle b_i, b_j \rangle d\mathrm{v}.$$

Then $\mathbf{J}$ induces an complex structure on V such that the inner product is $\mathbf{J}$-invariant. We introduce a distance in Grassmanian $G(2n, k)$ as follows,

$$\mathrm{d}(\Lambda_1, \Lambda_2)^2 = \sum_j \|pr^{\perp}_{\Lambda_2}(e_j)\|^2_{L^2} \tag{8.19}$$

for any two k-dimensional subspaces Λ_1, Λ_2 in $\mathbb{R}^{2n}$, where e_i is an unit orthogonal basis of Λ_1 and $pr^{\perp}_{\Lambda_2}$ is the compliment of orthogonal projection to Λ_2. First we suppose that

$$\mathrm{d}(W, \mathbf{J}W)^2 < \Psi,$$

where $W = span\{\nabla h_i | i = 1, 2, ..., 2n-2\}$. Then by Gram-Schmidt process, one can find a unit orthogonal basis w_i of W such that

$$\|\mathbf{J}w_i - w_{n-1+i}\|_{L^2} < \Psi.$$

It is equivalent to that there exists a matrix $a_{ij} \in GL(2n-2, \mathbb{R})$ which is nearly orthogonal such that

$$w_i = \Sigma_j a_{ij} \nabla h_j.$$

Thus by changing an orthogonal basis in $\mathbb{R}^{2n-2}$, (8.18) will be true.

Secondly, we suppose that

$$d(W, \mathbf{J}W) > \delta_0.$$

This implies that there exists some j such that

$$\|pr_W^{\perp}(\mathbf{J}\nabla h_i)\|_{L^2} = \|\mathbf{J}\nabla h_i - pr_W(\mathbf{J}\nabla h_i)\|_{L^2} > \frac{\delta_0}{2n}.$$

Let

$$X = \frac{pr_W^{\perp}(\mathbf{J}\nabla h_i)}{pr_W^{\perp}(\mathbf{J}\nabla h_i)} \tag{8.20}$$

Then $pr_W^{\perp}(\mathbf{J}\nabla h_i)$ is perpendicular to W with $\|pr_W^{\perp}(\mathbf{J}\nabla h_i)\|_{L^2} = 1$ and it satisfies the condition (8.1) in Lemma 8.1. Thus we see that there exists a f-harmonic function θ which satisfies the conditions (8.9), (8.10) and (8.11) in Lemma 8.2. As a consequence, $B_p(\frac{1}{8})$ will almost spilt off along a new line associated to the coordinate function θ. Since $X \in W^{\perp}$, $B_p(\frac{1}{8})$ in fact split off $\mathbb{R}^{2n-1}$ almost. But the late implies that $B_p(\frac{1}{8})$ is close to an Euclidean ball in Gromov-Hausdorff topology by using a topological argument as in Theorem 6.2 in [CC2] or by the following Proposition 8.5 for Kähler manifolds.

□

By using the similar argument in Proposition 8.4, we prove

Proposition 8.5. *Let Y be a limit space of a sequence of Kähler manifolds in Theorem 5.1. Then*

$$\mathcal{S}(Y) = \mathcal{S}_{2k+1} = \mathcal{S}_{2k}.$$

Proof. We suffice to show that if a tangent cone T_yY at a point $y \in Y$ can split off $\mathbb{R}^{2k+1}$, T_yY can split off $\mathbb{R}^{2k+2}$. Let h_i be $2k+1$ f-harmonic functions which approximate $2k+1$ distance functions with different directions as constructed in Section 2 and Section 3. Then as in the proof of Proposition 8.4, we consider a linear space $V = \text{span}\{\nabla h_i, \mathbf{J}\nabla h_i\}$ with L^2-inner product. Since the dimension of $W = \text{span}\{\nabla h_i\}$ is odd, we have

$$d(W, \mathbf{J}W) \geq 1.$$

Thus T_yY will split off a new line. The proposition is proved.

□

References

[BE] Bakry, D. and Emery M., Diffusions hypercontractives, In Séminaire de probabilités, XIX, 1983/84, Lecture Notes in Math., vol. 1123, 177-206, Springer, Berlin, 1985.

[C1] Cheeger, J., Differentiability of Lpischitz functions on metric measure spaces, GAFA Vol. 9 (1999), 428-517.

[C2] Cheeger, J., Degeneration of Riemannian metrics under Ricci curvature bounds, Scuola Normale Superiore, Pisa (2001).

[C3] Cheeger, J., Integral bound on curvature,epplictic estimates and rectifiability of singular sets , GAFA Vol. 13 (2003), 20-72.

[CC1] Cheeger, J. and Colding, T., Lower bounds on Ricci curvature and almost rigidity of warped product, Ann. of Math. 144 (1996), 189-237.

[CC2] Cheeger, J. and Colding T., On the structure of spaces with Ricci curvature bounded below I, J. Differential Geom. 45 (1997), 406-480.

[CC3] Cheeger, J. and Colding, T., On the structure of spaces with Ricci curvature bounded below II, J. Differential Geom. 54 (2000), 13-35.

[CCT] Cheeger, J. , Colding, T. and Tian, G., On the singularities of spaces with bounded Ricci curvature, GAFA Vol. 12 (2002), 873-914.

[Co] Colding, T. , Ricci curvature and the volume convergence, Ann. of Math. 145 (1997), 477-504.

[Fu] Futaki, A., Kähler-Einstein metrics and integral invariants, Lecture notes in Math, vol. 1314 (1988), Springer-Verlag, Berlin, New-York.

[Gr] Gromov, M., Structures métriques pour les variétés riemanniennes. Edited by J. Lafontaine and P. Pansu. Textes Mathématiques, 1. CEDIC, Paris, 1981.

[Ha] Hamilton, R. S., The formation of singularities in the Ricci flow, Surv. Diff. Geom., vol. 2 (1995), 7-136, International Press.

[Ma] Mabuchi, T., Multiplier hermitian structures on Kähler manifolds, Nagoya Math. J. 170 (2003), 73-115.

[SY] Scheon, R. and Yau, S.T., Lectures on Differential Geometry, Conf. Proc. and Lecture Notes in Geometry and Topology, vol. 1 (1994), International Press.

[TZ1] Tian, G. and Zhu, X.H., Uniqueness of Kähler-Ricci soltions, Acta Math., 184 (2000), 271-305.

[TZ2] Tian, G. and Zhu, X.H., A new holomorphic invariant and uniqueness of Kähler-Ricci solitons, Comment. Math. Helv. 77 (2002), 297-325.

[TZh] Tian, G. and Zhang, Z., Degeneration of Kähler Ricci solitons, Intern. Math. Res. Notices, 2012, 957-985.

[TW] Tian, G. and Wang, B., On the structure of almost Einstein manifolds, arxiv.1202.2912.

[WW] Wei, G. and Wylie, W., Comparison geometry for Bakry-Emery Ricci curvature, J. Differential Geom. 83 (2009), 337-405.

[Zh] Zhu, X.H., Kähler-Ricci soliton type equations on compact complex manifolds with $C_1(M) > 0$, J. Geom. Anal., 10 (2000), 759-774.

Feng Wang, School of Mathematical Sciences, Peking University, Beijing, 100871, China

Xiaohua Zhu, School of Mathematical Sciences and BICMR, Peking University, Beijing, 100871, China, xhzhu@math.pku.edu.cn